1. Consider a three dimensional ket space. If a certain set of orthonormal kets, say $|\,1\,\rangle$, $|\,2\,\rangle$ and $|\,3\,\rangle$ are used as the base kets, then the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

in which both a and b are real.

a) Obviously A exhibits a degenerate spectrum. Does B have a degenerate spectrum as well?

b) Show that A and B commute.

c) Find a new (orthonormal) set of base kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

Solution:

a) From the matrix representation of B we can see that the ket $|\,1\,\rangle$ is an eigenvector of operator B with eigenvalue b, *i.e.*

$$B\,|\,1\,\rangle = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} = b \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = b\,|\,1\,\rangle\,.$$

As the matrix of operator B is Hermitian so its eigenvalues must be real. It only remains to diagonalize the minor M_{11} of matrix B.

$$\det\left(M_{11} - \lambda I\right) = 0,$$

therefore

$$\lambda^2 + b^2 = 0 \quad \rightarrow \quad \lambda = \pm b.$$

We have found that the eigenvalues of B are $\{-b, b, b\}$, concluding that operator B has a degenerate spectrum.

b) Let us calculate the products AB and BA independently.

$$AB = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix},$$

$$BA = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}.$$

We are now in conditions of writing the commutator:

$$[A, B] = AB - BA = 0.$$

Therefore A and B must share a simultaneous set of eigenvectors.

c) We already have the first of the eigenvector in that particular set, *i.e.* ket $|\,1\,\rangle$. Let us find now the remaining eigenvectors of operator B in the subspace M_{11}.

- Eigenvector associated to eigenvalue b. Let us rename it as $|\,2'\,\rangle$.

$$\begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix} \begin{pmatrix} c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} bc_2 \\ bc_3 \end{pmatrix}$$

  Thus

$$\begin{cases} -ibc_3 & = bc_2 \\ \quad ibc_2 & = bc_3 \end{cases} \rightarrow c_3 = ic_2.$$

  If we want our eigenvectors normalized, then $c_2^2 = c_3^2 = 1/2$.

- Eigenvector associated to eigenvalue $-b$. Let us rename it as $|\,3'\,\rangle$.

$$\begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix} \begin{pmatrix} c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -bc_2 \\ -bc_3 \end{pmatrix}$$

  Thus

$$\begin{cases} ibc_3 & = bc_2 \\ ibc_2 & = -bc_3 \end{cases} \rightarrow c_3 = -ic_2.$$

  If we want our eigenvectors normalized, then $c_2^2 = c_3^2 = 1/2$.

We have to check that these new eigenvectors are shared with operator A.

$$\begin{aligned} A\,|\,2'\,\rangle &= \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -a \\ -ia \end{pmatrix} = -a\,|\,2'\,\rangle \\ A\,|\,3'\,\rangle &= \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -a \\ ia \end{pmatrix} = -a\,|\,3'\,\rangle \end{aligned}$$

The primed notation ($|1'\rangle$) is enough, but another and sure more informatic naming convecntion is to characterize the eigenvectors with their eigenvalues respect operator A and B, respectively

$$|a, b\rangle = |1\rangle \qquad |-a, b\rangle = \frac{1}{\sqrt{2}}(|2\rangle + \mathrm{i}\,|3\rangle) \qquad |-a, -b\rangle = \frac{1}{\sqrt{2}}(|2\rangle - \mathrm{i}\,|3\rangle)$$

2. Evaluate the uncertainty relation of x and p operators for a particle confined in an infinite potential well (between two unpenetrable walls.) Some help: In this case the potential can be written:$V(x) = 0$, when $0 < x < a$ and otherwise $V = \infty$. From quantum mechanics we remember that the wave function in such a potential reads $\psi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$, in which number n refers to the nth excitation while $n = 1$ is the ground state.

Solution: The uncertainty relation for x and p is given by the product of the standard deviations Δx and Δp, *i.e.*, $\Delta x \Delta p$. The standard deviation for a generic observable q in the system state ψ is given by

$$\Delta_\psi q = \sqrt{\langle q^2\rangle_\psi - \langle q\rangle_\psi^2}.$$

In our case, we know that for a potential well in which $V = 0$ in the range $0 < x < a$, the eigenfunctions for this particular problem can be written as

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right).$$

Knowing this, we are ready to calculate the expectation values of observables x, x^2, p and p^2.

$$\begin{aligned}
\langle x\rangle &= \int \psi_n^*(x)x\psi_n(x)\,\mathrm{d}x = \int_0^a \frac{2}{a}x\sin^2(n\pi x/a)\,\mathrm{d}x \\
&= \frac{2}{a}\frac{a^2}{4} = \frac{a}{2}, \\
\langle x^2\rangle &= \int \psi_n^*(x)x^2\psi_n(x)\,\mathrm{d}x = \int_0^a \frac{2}{a}x^2\sin^2(n\pi x/a)\,\mathrm{d}x \\
&= \frac{2}{a}\frac{a^3}{6}\left(1-\frac{3}{2\pi^2n^2}\right) = \frac{a^2}{3}\left(1-\frac{3}{2\pi^2n^2}\right), \\
\langle p\rangle &= \int \psi_n^*(x)(-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x}\psi_n(x)\,\mathrm{d}x \\
&= -i\hbar\frac{2}{a}\int_0^a \sin(n\pi x/a)n\pi/a\cos(n\pi x/a)\,\mathrm{d}x = 0, \\
\langle p^2\rangle &= \int \psi_n^*(x)(-\hbar^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_n(x)\,\mathrm{d}x = \hbar^2\frac{2}{a}\frac{n^2\pi^2}{a^2}\int_0^a \sin^2(n\pi x/a)\,\mathrm{d}x \\
&= \hbar^2\frac{2}{a}\frac{n^2\pi^2}{a^2}\frac{a}{2} = \hbar^2\pi^2n^2/a^2
\end{aligned}$$

In the evaluation one needs partial integration and double angle formula: $\sin^2 x = 1/2(1-\cos 2x)$. Subtituting the above values into the definition for the uncertainty relation, we obtain that

$$\Delta x\Delta p = \frac{\hbar}{2}\sqrt{\frac{\pi^2n^2-6}{3}} > \frac{\hbar}{2}\sqrt{\frac{3^2n^2-6}{3}} > \frac{\hbar}{2}\sqrt{\frac{9\cdot 1^2-6}{3}} = \frac{\hbar}{2}.$$

3. Show that

$$\langle p' | x | \alpha \rangle = \mathrm{i}\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle \qquad \text{and} \qquad \langle \beta | x | \alpha \rangle = \int \mathrm{d}p' \, \phi_\beta^\star(p') \mathrm{i}\hbar \frac{\partial}{\partial p'} \phi_\alpha(p').$$

Solution: Here we need the representation of eigenstate $|x'\rangle$ in the momentum space

$$\langle p' | x' \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\mathrm{i} \frac{p' x'}{\hbar} \right),$$

also we need the hermicity of x ($\langle x' | x | \alpha \rangle = x' \langle x' | \alpha \rangle$) and the familiar differentation rule $\partial(\exp(ax))\partial x = a \exp(ax)$.

$$\begin{aligned}
\langle p' | x | \alpha \rangle = \left\langle p' \left| \underbrace{\int \mathrm{d}x' \; |x'\rangle \langle x'|}_{I} \, x \right| \alpha \right\rangle &= \int \mathrm{d}x' \, x' \langle p' | x' \rangle \langle x' | \alpha \rangle \\
&= \int \mathrm{d}x' \, x' \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\mathrm{i} \frac{p' x'}{\hbar} \right) \langle x' | \alpha \rangle \\
&= \int \mathrm{d}x' \, \mathrm{i}\hbar \frac{\partial}{\partial p'} \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\mathrm{i} \frac{p' x'}{\hbar} \right) \langle x' | \alpha \rangle \\
&= \mathrm{i}\hbar \frac{\partial}{\partial p'} \left\langle p' \left| \int \mathrm{d}x' \; |x'\rangle \langle x'| \right| \alpha \right\rangle = \mathrm{i}\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle
\end{aligned}$$

The second result is a corollary of the first one:

$$\begin{aligned}
\langle \beta | x | \alpha \rangle = \left\langle \beta \left| \underbrace{\int \mathrm{d}p' \; |p'\rangle \langle p'|}_{I} \, x \right| \alpha \right\rangle &= \int \mathrm{d}p' \, \langle \beta | p' \rangle \langle p' | x | \alpha \rangle \\
&= \int \mathrm{d}p' \, \phi_\beta^\star(p') \mathrm{i}\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle \\
&= \int \mathrm{d}p' \, \phi_\beta^\star(p') \mathrm{i}\hbar \frac{\partial}{\partial p'} \phi_\alpha(p').
\end{aligned}$$

4. Consider spin precession of electron in static uniform magnetic field in the z direction and calculate the expectation values of spin at time t in y and z directions when the initial state of the system at $t = 0$ is

$$|S_x;\uparrow\rangle = \frac{1}{\sqrt{2}}|S_z;\uparrow\rangle + \frac{1}{\sqrt{2}}|S_z;\downarrow\rangle\,.$$

Solution:
We consider magnetic field to be $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, so that the Hamiltonian is written as

$$H = -\mu\boldsymbol{B}\boldsymbol{S} = -\left(\frac{eB}{m_e c}\right)S_z = \omega_c S_z.$$

The time evolution operator for this system is

$$\mathcal{U}(t,0) = \exp\left(-iS_z\omega_c t/\hbar\right),$$

where $\omega_c = |e|B/m_e c$. The expectation value for S_z is then calculated as follows

$$\begin{aligned}
\langle S_z\rangle(t) &= \langle S_x;\uparrow|\mathcal{U}^\dagger(t,0)S_z\mathcal{U}(t,0)|S_x,\uparrow\rangle \\
&= \left[e^{i\omega_c t/2}\langle S_z;\uparrow| + e^{-i\omega_c t/2}\langle S_z;\downarrow|\right]\frac{S_z}{2}\left[e^{-i\omega_c t/2}|S_z;\uparrow\rangle + e^{i\omega_c t/2}|S_z;\downarrow\rangle\right] \\
&= 0.
\end{aligned}$$

Calculating the expectation value of S_y is equivalent:

$$\begin{aligned}
\langle S_y\rangle(t) &= \langle S_x;\uparrow|\mathcal{U}^\dagger(t,0)S_y\mathcal{U}(t,0)|S_x,\uparrow\rangle \\
&= \left[e^{i\omega_c t/2}\langle S_z;\uparrow| + e^{-i\omega_c t/2}\langle S_z;\downarrow|\right]\frac{S_y}{2}\left[e^{-i\omega_c t/2}|S_z;\uparrow\rangle + e^{i\omega_c t/2}|S_z;\downarrow\rangle\right] \\
&= \left[e^{i\omega_c t/2}\langle S_z;\uparrow| + e^{-i\omega_c t/2}\langle S_z;\downarrow|\right]i\frac{\hbar}{4}\left[e^{-i\omega_c t/2}|S_z;\downarrow\rangle - e^{i\omega_c t/2}|S_z;\uparrow\rangle\right] \\
&= \frac{\hbar}{2}\sin\left(\omega_c t\right).
\end{aligned}$$

It is trivial to see that spin precedes in the xy-plane with a frequency ω_c and with no projection into the z axis.
Another and more straightforward way to calculate the expectation values is to apply the matrix representations of S_i derived in previous exercises and represent also the time dependent state in the eigenbasis of S_z.

$$|S_x;\uparrow;t\rangle = \mathcal{U}(t,0)\,|S_x;\uparrow;t=0\rangle = \begin{pmatrix}\frac{1}{2}e^{-i\omega_c t/2}\\ \frac{1}{2}e^{i\omega_c t/2}\end{pmatrix}$$

$$\langle S_x\rangle(t) = \langle S_x;\uparrow;t\,|S_z|\,S_x;\uparrow;t\rangle = \frac{\hbar}{2}\begin{pmatrix}\frac{1}{2}e^{i\omega_c t/2} & \frac{1}{2}e^{-i\omega_c t/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{2}e^{-i\omega_c t/2}\\ \frac{1}{2}e^{i\omega_c t/2}\end{pmatrix} = 0$$

$$\begin{aligned}
\langle S_y\rangle(t) = \langle S_x;\uparrow;t\,|S_z|\,S_x;\uparrow;t\rangle &= \frac{\hbar}{2}\begin{pmatrix}\frac{1}{2}e^{i\omega_c t/2} & \frac{1}{2}e^{-i\omega_c t/2}\end{pmatrix}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}e^{-i\omega_c t/2}\\ \frac{1}{2}e^{i\omega_c t/2}\end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix}\frac{1}{2}e^{i\omega_c t/2} & \frac{1}{2}e^{-i\omega_c t/2}\end{pmatrix}\begin{pmatrix}-\frac{\mathrm{i}}{2}e^{i\omega_c t/2}\\ \frac{\mathrm{i}}{2}e^{-i\omega_c t/2}\end{pmatrix} = \frac{\hbar}{2}\sin(\omega_c t)
\end{aligned}$$