1 Single particle states

We work with parabolically confined dots affected by spin-orbit (SO) coupling and subjected to external radiation (and static magnetic) fields. Classically, the Lorentz force acting on a charged particle in a magnetic field is given by

$$m\ddot{\vec{x}} = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}. \tag{1}$$

In order to account for this magnetic field in the Hamiltonian, a vector potential $\vec{A}$ must be introduced. This potential is composed of the static part $\vec{A}_B$ and the radiative part A_E, as

$$\vec{A} = \vec{A}_B + \vec{A}_E = \frac{B}{2}(-y, x, 0) - \frac{c\vec{E}}{\Omega}\sin\Omega t. \tag{2}$$

The magnetic and radiation fields are related to these vector potentials by

$$\vec{B} \;=\; \nabla \times \vec{A} = \nabla \times \vec{A}_B \tag{3}$$

$$\vec{E}\cos\Omega t \;=\; -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}_E. \tag{4}$$

The Lagrangian for (80) is given by

$$\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}m\dot{\vec{x}}^2 + \frac{e}{c}\dot{\vec{x}} \cdot \vec{A}(\vec{x}). \tag{5}$$

From this the momentum canonically conjugate to x_j is found to be

$$p_j \equiv \frac{\partial \mathcal{L}}{\partial \dot{x_j}} = m\dot{x_j} + \frac{e}{c}A_j \tag{6}$$

and taking the time derivative gives

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_j}}\right) = m\ddot{x_j} + \frac{e}{c}\frac{dA_j}{dt} = m\ddot{x_j} + \frac{e}{c}\sum_{k=1}^{3}\dot{x_k}\frac{\partial A_j}{\partial x_k}. \tag{7}$$

In order to apply the Euler-Lagrange equation, it is also necessary to calculate

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{e}{c}\sum_k x_k \frac{\partial A_k}{\partial x_j}. \tag{8}$$

Now using these equations in the Euler-Lagrange equation gives

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_j}}\right) - \frac{\partial \mathcal{L}}{\partial \dot{x_j}} = m\ddot{x_j} + \frac{e}{c}\sum_{k=1}^{3}\dot{x_k}\frac{\partial A_j}{\partial x_k} - \frac{e}{c}\sum_{k=1}^{3}x_k\frac{\partial A_k}{\partial x_j}, \tag{9}$$

which can be rearranged as

$$m\ddot{x_j} = \frac{e}{c}\sum_{k=1}^{3}\dot{x_k}\left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}\right) = \frac{e}{c}\sum_{k=1}^{3}\varepsilon_{jkl}\dot{x_k}B_l. \tag{10}$$

The Hamiltonian accounting for this magnetic field is now given by

$$\begin{aligned}\mathcal{H}(\vec{x},\vec{p}) &= \dot{\vec{x}}\cdot\vec{p}-\mathcal{L}\\ &= \dot{\vec{x}}\cdot\left(\frac{e}{c}\vec{A}+m\dot{\vec{x}}\right)-\frac{1}{2}m\dot{\vec{x}}^2-\frac{e}{c}\dot{\vec{x}}\cdot\vec{A}\\ &= \frac{1}{2}m\dot{\vec{x}}^2=\frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2\end{aligned}$$

The total single particle Hamiltonian is then

$$H_{\mathrm{SP}}=\frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2+\frac{1}{2}m\omega_0^2r^2+\frac{\alpha}{\hbar}\left[\vec{\sigma}\times\left(\vec{p}-\frac{e}{c}\vec{A}\right)\right]_z+\frac{1}{2}g\mu_BB\sigma_z. \quad (11)$$

Here $\vec{\sigma}$ is the vector of Pauli matrices, i.e.

$$\vec{\sigma}=\sigma_x\vec{i}+\sigma_y\vec{j}+\sigma_z\vec{k}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\vec{i}+\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}\vec{j}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\vec{k}, \quad (12)$$

Furthermore, in the Hamiltonian (89) the parameter m is the effective mass of the particle, ω_0 the angular frequency of the parabolic confinement, α the strength of the Rashba SO coupling, e the charge of the particle (so, for electrons $e=-|e|$) and g the effective Lande gyromagnetic ratio.

We now separate in(89) the radiative terms from the static ones. For that purpose we introduce the kinematic momentum

$$\vec{\Pi}=\vec{p}-\frac{e}{c}\vec{A}_B, \quad (13)$$

which allows us to write

$$\begin{aligned}H_{\mathrm{SP}} &= \frac{1}{2m}\left(\vec{\Pi}-\frac{e}{c}\vec{A}_E\right)^2+\frac{1}{2}m\omega_0^2r^2+\frac{\alpha}{\hbar}\left[\vec{\sigma}\times\left(\vec{\Pi}-\frac{e}{c}\vec{A}_E\right)\right]_z+\frac{1}{2}g\mu_BB\sigma_z\\ &= \frac{1}{2m}\Pi^2+\frac{1}{2}m\omega_0^2r^2-+\frac{\alpha}{\hbar}\left[\vec{\sigma}\times\vec{\Pi}\right]_z+\frac{1}{2}g\mu_BB\sigma_z\\ &\quad -\frac{e}{mc}\vec{A}_E\cdot\vec{\Pi}+\frac{e^2}{2mc^2}A_E^2-\frac{\alpha e}{\hbar c}\left[\vec{\sigma}\times\vec{A}_E\right]_z\\ &= H_{\mathrm{S}}+H_{\mathrm{R}}.\end{aligned}$$

Here H_{S} is the static Hamiltonian

$$H_{\mathrm{S}}=\frac{1}{2m}\Pi^2+\frac{1}{2}m\omega_0^2r^2+\frac{\alpha}{\hbar}\left[\vec{\sigma}\times\vec{\Pi}\right]_z+\frac{1}{2}g\mu_BB\sigma_z \quad (14)$$

and H_{R} the radiative Hamiltonian

$$H_{\mathrm{R}}=-\frac{e}{mc}\vec{A}_E\cdot\vec{\Pi}+\frac{e^2}{2mc^2}A_E^2-\frac{\alpha e}{\hbar c}\left[\vec{\sigma}\times\vec{A}_E\right]_z. \quad (15)$$

1.1 Static Hamiltonian without SO coupling

We now solve the static single particle problem in the case where the SO coupling vanishes. Our Hamiltonian is then simply

$$H_{\mathrm{S}}^{0} = \frac{1}{2m}\Pi^2 + \frac{1}{2}m\omega_0^2 r^2 + \frac{1}{2}g\mu_B B\sigma_z. \tag{16}$$

Recalling that the vector potential was

$$\vec{A}_B = \frac{B}{2}(-y, x, 0)$$

the operator Π^2 can be written as

$$\begin{aligned}
\Pi^2 &= \left(\vec{p} - \frac{e}{c}\vec{A}_B\right)^2 = p^2 - \frac{2e}{c}\vec{A}_B \cdot \vec{p} + \frac{e^2}{c^2}A_B^2 \\
&= -\hbar^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \mathrm{i}\frac{2e\hbar}{c}\frac{B}{2}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + \frac{e^2}{c^2}\frac{B^2}{4}(y^2 + x^2).
\end{aligned}$$

We transform to polar coordinates

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}$$

using the relations

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\
\frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y},
\end{aligned}$$

or inversely

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{17}$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}, \tag{18}$$

which yield

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \\
&\quad + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} - 2\frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \\
&\quad + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} &= \left(\sin\theta\,\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\right)\left(\sin\theta\,\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\right)\\
&= \sin^2\theta\,\frac{\partial^2}{\partial r^2}-\frac{\sin\theta\cos\theta}{r^2}\,\frac{\partial}{\partial\theta}+\frac{\sin\theta\cos\theta}{r}\,\frac{\partial^2}{\partial r\partial\theta}\\
&\quad+\frac{\cos^2\theta}{r}\,\frac{\partial}{\partial r}+\frac{\cos\theta\sin\theta}{r}\,\frac{\partial^2}{\partial\theta\partial r}-\frac{\cos\theta\sin\theta}{r^2}\,\frac{\partial}{\partial\theta}+\frac{\cos^2\theta}{r^2}\,\frac{\partial^2}{\partial\theta^2}\\
&= \sin^2\theta\,\frac{\partial^2}{\partial r^2}+2\,\frac{\cos\theta\sin\theta}{r}\,\frac{\partial^2}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\,\frac{\partial^2}{\partial\theta^2}\\
&\quad+\frac{\cos^2\theta}{r}\,\frac{\partial}{\partial r}-2\,\frac{\sin\theta\cos\theta}{r^2}\,\frac{\partial}{\partial\theta}
\end{aligned}$$

Thus in polar coordinates the ∇^2 operator takes the form

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\,\frac{\partial}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}.$$

We also need to evaluate the term

$$\begin{aligned}
-y\,\frac{\partial}{\partial x}+x\,\frac{\partial}{\partial y} &= -r\sin\theta\left(\cos\theta\,\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right)\\
&\quad+r\cos\theta\left(\sin\theta\,\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\right)\\
&= -r\sin\theta\cos\theta\,\frac{\partial}{\partial r}+\sin^2\theta\,\frac{\partial}{\partial\theta}\\
&\quad+r\cos\theta\sin\theta\,\frac{\partial}{\partial r}+\cos^2\theta\,\frac{\partial}{\partial\theta}\\
&= \frac{\partial}{\partial\theta}.
\end{aligned}$$

The term Π^2 can now be written in polar coordinates as

$$\Pi^2=-\hbar^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\,\frac{\partial}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}\right)+\mathrm{i}\,\frac{eB\hbar}{c}\,\frac{\partial}{\partial\theta}+\frac{e^2B^2}{4c^2}\,r^2.$$

The static Hamiltonian (94) is now

$$\begin{aligned}
H^0_{\mathrm{S}} &= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\,\frac{\partial}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}\right)+\mathrm{i}\,\frac{eB\hbar}{2mc}\,\frac{\partial}{\partial\theta}+\frac{e^2B^2}{8mc^2}\,r^2+\frac{1}{2}m\omega_0^2r^2\\
&\quad+\frac{1}{2}g\mu_B B\sigma_z.
\end{aligned}$$

We define the cyclotron frequency ω_c as

$$\omega_c=\frac{eB}{mc} \tag{19}$$

and substitute it to the Hamiltonian to get

$$\begin{aligned}H_{\mathrm{S}}^{0} &= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)+\mathrm{i}\,\frac{\hbar\omega_c}{2}\frac{\partial}{\partial\theta}+\frac{m\omega_c^2}{8}r^2+\frac{1}{2}m\omega_0^2r^2\\&\quad+\frac{1}{2}g\mu_B B\sigma_z\\&= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)+\frac{\mathrm{i}}{2}\hbar\omega_c\frac{\partial}{\partial\theta}+\frac{1}{2}m\left(\frac{\omega_c^2}{4}+\omega_0^2\right)r^2\\&\quad+\frac{1}{2}g\mu_B B\sigma_z\end{aligned}$$

Defining the effective frequency ω as

$$\omega^2=\omega_0^2+\frac{1}{4}\omega_c^2 \tag{20}$$

we end up with

$$H_{\mathrm{S}}^{0}=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)+\frac{\mathrm{i}}{2}\hbar\omega_c\frac{\partial}{\partial\theta}+\frac{1}{2}m\omega^2r^2+\frac{1}{2}g\mu_B B\sigma_z.$$

If we now change to variable

$$s=\frac{r}{a}$$

we get

$$H_{\mathrm{S}}^{0}=-\frac{\hbar^2}{2ma^2}\left(\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}+\frac{1}{s^2}\frac{\partial^2}{\partial\theta^2}\right)+\frac{\mathrm{i}}{2}\hbar\omega_c\frac{\partial}{\partial\theta}+\frac{1}{2}m\omega^2a^2s^2+\frac{1}{2}g\mu_B B\sigma_z.$$

Requiring that

$$\frac{\hbar^2}{2ma^2}=\frac{1}{2}m\omega^2a^2,$$

or

$$a^2=\frac{\hbar}{m\omega} \tag{21}$$

we get

$$H_{\mathrm{S}}^{0}=-\frac{1}{2}\hbar\omega\left(\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}-s^2\right)-\frac{\hbar\omega}{2s^2}\frac{\partial^2}{\partial\theta^2}+\frac{\mathrm{i}}{2}\hbar\omega_c\frac{\partial}{\partial\theta}+\frac{1}{2}g\mu_B B\sigma_z. \tag{22}$$

Because the system is cylindrically symmetric we can write the eigenfunctions ψ of H_{S}^{0} in the form

$$\psi(s,\theta)=\varphi(s)e^{im\theta}, \tag{23}$$

where m can take only integer values. Substituting this into the Schrödinger equation

$$H_{\mathrm{S}}^{0}\psi=\varepsilon\psi, \tag{24}$$

we have

$$-\varphi'' - \frac{1}{s}\varphi' + s^2\varphi + \frac{m^2}{s^2}\,\varphi - m\,\frac{\omega_c}{\omega}\varphi \pm \frac{g\mu_B B}{\hbar\omega}\varphi = \frac{2\varepsilon}{\hbar\omega}\,\varphi,$$

where the sign of the factor $\pm\frac{1}{2}$ in the Zeeman term is to be chosen according to the spin.

Once again we change to a new variable $z = s^2$. Using the notation

$$\phi(z) = \varphi(s)$$

we have

$$\varphi'(s) = \phi'(z)z' = 2s\phi'(z) = 2\sqrt{z}\phi'(z)$$

and

$$\varphi''(s) = 2(\sqrt{z}\phi'(z))'z' = 2\left(\frac{1}{2\sqrt{z}}\phi'(z) + \sqrt{z}\phi''z\right)2\sqrt{z} = 2\phi'(z) + 4z\phi''(z).$$

The Schrödinger equation reads now

$$-4z\phi'' - 2\phi' - \frac{1}{\sqrt{z}}2\sqrt{z}\phi' + z\phi + \frac{m^2}{z}\,\phi - m\,\frac{\omega_c}{\omega}\phi \pm \frac{g\mu_B B}{\hbar\omega}\phi = \frac{2\varepsilon}{\hbar\omega}\phi,$$

or

$$z\phi'' + \phi' + \left(\nu - \frac{m^2}{4z} - \frac{z}{4} + \frac{m\kappa}{2} \mp \gamma\right)\phi = 0, \tag{25}$$

where we have set

$$\nu = \frac{\varepsilon}{2\hbar\omega} \tag{26}$$

$$\kappa = \frac{\omega_c}{2\omega} \tag{27}$$

$$\gamma = \frac{g\mu_B B}{4\hbar\omega}. \tag{28}$$

The signs in front of γ correspond to spin up and down states, respectively.

To solve the differential equation we substitute

$$\phi = \mathrm{e}^{-\alpha z} z^\beta \xi(z)$$

for ϕ. We evaluate the derivatives of ϕ as

$$\phi' = \left(-\alpha z^\beta\xi + \beta z^{\beta-1}\xi + z^\beta\xi'\right)\mathrm{e}^{-\alpha z} = \mathrm{e}^{-\alpha z} z^{\beta-1}\left[z\xi' + (-\alpha z + \beta)\xi\right]$$

and

$$\begin{aligned}
\phi'' = &\\
&\left(-\alpha\beta z^{\beta-1}\xi - \alpha z^\beta\xi' + \beta(\beta-1)z^{\beta-2}\xi + \beta z^{\beta-1}\xi' + \beta z^{\beta-1}\xi' + z^\beta\xi''\right)\mathrm{e}^{-\alpha z}\\
&+ \left(\alpha^2 z^\beta\xi - \alpha\beta z^{\beta-1}\xi - \alpha z^\beta\xi'\right)\mathrm{e}^{-\alpha z} =\\
&= \mathrm{e}^{-\alpha z}\left[z^\beta\xi'' + \left(-2\alpha z^\beta + 2\beta z^{\beta-1}\right)\xi'\right.\\
&\left.+ \left(-2\alpha\beta z^{\beta-1} + \beta(\beta-1)z^{\beta-2} + \alpha^2 z^\beta\right)\xi\right]\\
&= \mathrm{e}^{-\alpha z} z^{\beta-2}\left[z^2\xi'' + 2\left(-\alpha z^2 + \beta z\right)\xi' + \left(-2\alpha\beta z + \beta(\beta-1) + \alpha^2 z^2\right)\xi\right]
\end{aligned}$$

The first two terms in the differential eq. (103) are now

$$
\begin{aligned}
& z\phi'' + \phi' = \\
& \quad \mathrm{e}^{-\alpha z} z^{\beta-1}\left[z^2\xi'' + \left(-2\alpha z^2 + (2\beta+1)z\right)\xi' + \left((-2\alpha\beta - \alpha)z + \beta^2 + \alpha^2 z^2\right)\xi\right] \\
& \quad = \mathrm{e}^{-\alpha z} z^{\beta}\left[z\xi'' + (-2\alpha z + 2\beta + 1)\,\xi' + \left(\alpha^2 z - 2\alpha\beta - \alpha + \frac{\beta^2}{z}\right)\xi\right]
\end{aligned}
$$

and the differential equation itself reads

$$
\begin{aligned}
z\xi'' &+ (-2\alpha z + 2\beta + 1)\,\xi' + \left(\alpha^2 z - 2\alpha\beta - \alpha + \frac{\beta^2}{z}\right)\xi \\
&+ \left(\nu - \frac{m^2}{4z} - \frac{z}{4} + \frac{m\kappa}{2} \mp \gamma\right)\xi = \\
z\xi'' &+ (-2\alpha z + 2\beta + 1)\,\xi' \\
&+ \left(\nu + \frac{m\kappa}{2} \mp \gamma - 2\alpha\beta - \alpha + \left(\alpha^2 - \frac{1}{4}\right)z + \frac{\beta^2 - \frac{m^2}{4}}{z}\right)\xi = 0.
\end{aligned}
$$

When we now set

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \left|\frac{m}{2}\right|
\end{aligned}
$$

we end up with

$$
z\xi'' + (|m| + 1 - z)\xi' + \left(\nu + \frac{m\kappa}{2} \mp \gamma - \frac{|m|}{2} - \frac{1}{2}\right)\xi = 0.
$$

This is of the form of the Laguerre differential equation

$$
zy'' + (q + 1 - z)y' + ny = 0
$$

which has a regular solution

$$
y = L_n^q(z)
$$

provided that n is a nonnegative integer. In our case this means that we have to have

$$
\nu = n - \frac{m\kappa}{2} + \frac{|m| + 1}{2} \pm \gamma;\ n = 0, 1, 2, \ldots
$$

and

$$
\xi(z) = L_n^{|m|}(z).
$$

Recalling that $\xi(z)$ was defined as

$$
\phi(z) = \mathrm{e}^{-z/2} z^{|m|/2} \xi(z)
$$

in terms of the wavefunction ϕ in the Schrödinger equation (103) and ϕ was related to φ via

$$
\varphi(s) = \phi(s^2) = \mathrm{e}^{-s^2/2} s^{|m|} L_n^{|m|}(s^2),
$$

which in turn was the radial part of the total wavefunction

$$\psi(s,\theta)=\varphi(s)\mathrm{e}^{\mathrm{i}m\theta},$$

we can write the eigenfunctions of the original static Hamiltonian H_{S}^{0} (94) in terms of original dimensional units $r=as$ like

$$\psi(\vec{r})=\mathrm{e}^{-r^2/2a^2}\left(\frac{r}{a}\right)^{|m|}L_n^{|m|}\left(\frac{r^2}{a^2}\right)\mathrm{e}^{\mathrm{i}m\theta}.$$

We now normalize this by evaluating

$$\begin{aligned}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty}r\,\mathrm{d}r\,|\psi(\vec{r})|^2 &= 2\pi\int_0^{\infty}r\,\mathrm{d}r\,\mathrm{e}^{-r^2/a^2}\left(\frac{r}{a}\right)^{2|m|}\left[L_n^{|m|}\left(\frac{r^2}{a^2}\right)\right]^2\\ &= 2\pi a^2\int_0^{\infty}s\,\mathrm{d}s\,\mathrm{e}^{-s^2}s^{2|m|}\left[L_n^{|m|}(s^2)\right]^2\\ &= \pi a^2\int_0^{\infty}\mathrm{d}z\,e^{-z}z^{|m|}\left[L_n^{|m|}(z)\right]^2\\ &= \pi a^2\frac{(n+|m|)!}{n!}\end{aligned}$$

which allows us to write the normalized wavefunctions as

$$\begin{aligned}\psi_{nm}(\vec{r}) &= \frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+|m|)!}}\mathrm{e}^{-r^2/2a^2}\left(\frac{r}{a}\right)^{|m|}L_n^{|m|}\left(\frac{r^2}{a^2}\right)\mathrm{e}^{\mathrm{i}m\theta}\\ &= \frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+|m|)!}}\mathrm{e}^{-s^2/2}s^{|m|}L_n^{|m|}(s^2)\mathrm{e}^{\mathrm{i}m\theta}\\ &= \frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+|m|)!}}\mathrm{e}^{-z/2}z^{|m|/2}L_n^{|m|}(z)\mathrm{e}^{\mathrm{i}m\theta}.\end{aligned}$$

Collecting everything together we have seen that the eigenfunctions and eigenvalues of the static Hamiltonian (94)

$$H_{\mathrm{S}}^{0}=\frac{1}{2m}\Pi^2+\frac{1}{2}m\omega_0^2r^2+\frac{1}{2}g\mu_B B\sigma_z$$

are

$$\psi_{nm}(r,\theta) = g_{nm}\left(\frac{r^2}{a^2}\right)e^{\mathrm{i}m\theta} \tag{29}$$

$$g_{nm}(z) = \frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+|m|)!}}\mathrm{e}^{-z/2}z^{|m|/2}L_n^{|m|}(z) \tag{30}$$

$$\varepsilon = 2\hbar\omega\nu \tag{31}$$

$$\nu = n-\frac{1}{2}m\kappa+\frac{1}{2}(|m|+1)\pm\gamma;\ n=0,1,2,\ldots \tag{32}$$

$$a^2 = \frac{\hbar}{m\omega} \tag{33}$$
$$\kappa = \frac{\omega_c}{2\omega} \tag{34}$$
$$\gamma = \frac{g\mu_B B}{4\hbar\omega} \tag{35}$$
$$\omega^2 = \omega_0^2 + \frac{1}{4}\,\omega_c^2 \tag{36}$$
$$\omega_c = \frac{eB}{mc}. \tag{37}$$

1.2 Rashba SO coupling

We will now concentrate on the term

$$H_{\mathrm{SO}} = \frac{\alpha}{\hbar}\left[\vec{\sigma}\times\left(\vec{p}-\frac{e}{c}\vec{A}\right)\right]_z = \frac{\alpha}{\hbar}\left[\vec{\sigma}\times\vec{\Pi}\right]_z \tag{38}$$

in the complete single particle Hamiltonian H_{SP} (89). We rewrite this componentwise as

$$\begin{aligned} H_{\mathrm{SO}} &= \frac{\alpha}{\hbar}\left[\vec{\sigma}\times\vec{\Pi}\right]_z = \frac{\alpha}{\hbar}\left[\sigma_x\Pi_y - \sigma_y\Pi_x\right] \\ &= \frac{\alpha}{\hbar}\left[\begin{pmatrix} 0 & \Pi_y \\ \Pi_y & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\mathrm{i}\Pi_x \\ \mathrm{i}\Pi_x & 0 \end{pmatrix}\right] \\ &= \frac{\alpha}{\hbar}\begin{pmatrix} 0 & \Pi_y + \mathrm{i}\Pi_x \\ \Pi_y - \mathrm{i}\Pi_x & 0 \end{pmatrix} = \mathrm{i}\,\frac{\alpha}{\hbar}\begin{pmatrix} 0 & \Pi_x - \mathrm{i}\Pi_y \\ -\Pi_x - \mathrm{i}\Pi_y & 0 \end{pmatrix}. \end{aligned}$$

Defining the ladder operators $\Pi_\pm$ as

$$\Pi_\pm = \Pi_x \pm \mathrm{i}\Pi_y \tag{39}$$

we get

$$H_{\mathrm{SO}} = \mathrm{i}\,\frac{\alpha}{\hbar}\begin{pmatrix} 0 & \Pi_- \\ -\Pi_+ & 0 \end{pmatrix}. \tag{40}$$

This is clearly Hermitian, since $\Pi_\pm^\dagger = \Pi_\mp$ so that

$$H_{\mathrm{SO}}^\dagger = -\mathrm{i}\,\frac{\alpha}{\hbar}\begin{pmatrix} 0 & -\Pi_+^\dagger \\ \Pi_-^\dagger & 0 \end{pmatrix} = -\mathrm{i}\,\frac{\alpha}{\hbar}\begin{pmatrix} 0 & -\Pi_- \\ \Pi_+ & 0 \end{pmatrix} = \mathrm{i}\,\frac{\alpha}{\hbar}\begin{pmatrix} 0 & \Pi_- \\ -\Pi_+ & 0 \end{pmatrix}.$$

We now have a closer look at the ladder operators. Using the explicit expression

$$\vec{\Pi} = \vec{p} - \frac{e}{c}\vec{A} = \vec{p} - \frac{eB}{2c}(-y, x, 0)$$

yields

$$\begin{aligned} \Pi_+ &= p_x + \frac{eB}{2c}\,y + \mathrm{i}p_y - \mathrm{i}\,\frac{eB}{2c}\,x = p_x + \mathrm{i}p_y - \frac{eB}{2c}(\mathrm{i}x - y) \\ \Pi_- &= p_x + \frac{eB}{2c}\,y - \mathrm{i}p_y + \mathrm{i}\,\frac{eB}{2c}\,x = p_x - \mathrm{i}p_y + \frac{eB}{2c}(\mathrm{i}x + y). \end{aligned}$$

Further, in the coordinate representation we can write

$$p_x \pm \mathrm{i}p_y = -\mathrm{i}\hbar\left(\frac{\partial}{\partial x} \pm \mathrm{i}\,\frac{\partial}{\partial y}\right).$$

Using the explicit expressions (95-96)

$$\begin{aligned}
\frac{\partial}{\partial x} &= \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta} \\
\frac{\partial}{\partial y} &= \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta},
\end{aligned}$$

we have

$$\begin{aligned}
p_x \pm \mathrm{i}p_y &= -\mathrm{i}\hbar\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta} \pm \mathrm{i}\sin\theta\,\frac{\partial}{\partial r} \pm \mathrm{i}\frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\right) \\
&= -\mathrm{i}\hbar\left((\cos\theta \pm \mathrm{i}\sin\theta)\,\frac{\partial}{\partial r} + \frac{\pm\mathrm{i}\cos\theta - \sin\theta}{r}\,\frac{\partial}{\partial\theta}\right) \\
&= -\hbar\left(\mathrm{i}(\cos\theta \pm \mathrm{i}\sin\theta)\,\frac{\partial}{\partial r} + \frac{\mp\cos\theta - \mathrm{i}\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right) \\
&= -\hbar\left(\mathrm{i}(\cos\theta \pm \mathrm{i}\sin\theta)\,\frac{\partial}{\partial r} \mp \frac{\cos\theta \pm \mathrm{i}\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right) \\
&= -\hbar\mathrm{e}^{\pm\mathrm{i}\theta}\left(\mathrm{i}\,\frac{\partial}{\partial r} \mp \frac{1}{r}\,\frac{\partial}{\partial\theta}\right) \\
&= -\mathrm{i}\hbar\mathrm{e}^{\pm\mathrm{i}\theta}\left(\frac{\partial}{\partial r} \pm \mathrm{i}\frac{1}{r}\,\frac{\partial}{\partial\theta}\right).
\end{aligned}$$

The vector potential part $\mathrm{i}x \mp y$ takes the form

$$\mathrm{i}x \mp y = \mathrm{i}(x \pm \mathrm{i}y) = \mathrm{i}r\mathrm{e}^{\pm\mathrm{i}\theta}$$

which allows us to write

$$\begin{aligned}
\Pi_\pm &= p_x \pm \mathrm{i}p_y \mp \frac{eB}{2c}(\mathrm{i}x \mp y) \\
&= -\mathrm{i}\hbar\mathrm{e}^{\pm\mathrm{i}\theta}\left(\frac{\partial}{\partial r} \pm \mathrm{i}\frac{1}{r}\,\frac{\partial}{\partial\theta}\right) \mp \mathrm{i}\frac{eB}{2c}\,r\mathrm{e}^{\pm\mathrm{i}\theta} \\
&= -i\hbar\mathrm{e}^{\pm\mathrm{i}\theta}\left(\frac{\partial}{\partial r} \pm \mathrm{i}\frac{1}{r}\,\frac{\partial}{\partial\theta} \pm \frac{eB}{2\hbar c}\,r\right)
\end{aligned}$$

Recalling that the cyclotron frequency was defined (97) as $\omega_c = eB/mc$ we finally get

$$\Pi_\pm = -i\hbar\mathrm{e}^{\pm\mathrm{i}\theta}\left(\frac{\partial}{\partial r} \pm \mathrm{i}\frac{1}{r}\,\frac{\partial}{\partial\theta} \pm \frac{m\omega_c}{2\hbar}\,r\right) \tag{41}$$

We now transfer first to the dimensionless length

$$s = \frac{r}{a} \tag{42}$$

as

$$\begin{aligned}
\Pi_{\pm} &= -i\frac{\hbar}{a}\mathrm{e}^{\pm\mathrm{i}\theta}\left(\frac{\partial}{\partial s} \pm \mathrm{i}\frac{1}{s}\frac{\partial}{\partial\theta} \pm \frac{ma^2\omega_c}{2\hbar}\, s\right) \\
&= -i\frac{\hbar}{a}\mathrm{e}^{\pm\mathrm{i}\theta}\left(\frac{\partial}{\partial s} \pm \mathrm{i}\frac{1}{s}\frac{\partial}{\partial\theta} \pm \frac{\hbar}{m\omega}\frac{m\omega_c}{2\hbar}\, s\right) \\
&= -i\frac{\hbar}{a}\mathrm{e}^{\pm\mathrm{i}\theta}\left(\frac{\partial}{\partial s} \pm \mathrm{i}\frac{1}{s}\frac{\partial}{\partial\theta} \pm \kappa s\right),
\end{aligned}$$

where we have substituted $a^2 = \hbar/m\omega$ (99) and $\kappa = \omega_c/2\omega$. Next we replace s with

$$z = s^2 \tag{43}$$

and correspondingly $\partial/\partial s$ with

$$\frac{\partial}{\partial s} = \frac{\partial z}{\partial s}\frac{\partial}{\partial z} = 2s\,\frac{\partial}{\partial z} = 2\sqrt{z}\,\frac{\partial}{\partial z}, \tag{44}$$

which yields

$$\Pi_{\pm} = -i\frac{\hbar}{a}\mathrm{e}^{\pm\mathrm{i}\theta}\left(2\sqrt{z}\,\frac{\partial}{\partial z} \pm \mathrm{i}\frac{1}{\sqrt{z}}\frac{\partial}{\partial\theta} \pm \kappa\sqrt{z}\right),$$

or

$$\Pi_{\pm} = -i\frac{\hbar}{a}\mathrm{e}^{\pm\mathrm{i}\theta}\sqrt{z}\left(2\,\frac{\partial}{\partial z} \pm \mathrm{i}\frac{1}{z}\frac{\partial}{\partial\theta} \pm \kappa\right). \tag{45}$$

The next task is to find out the effects of the ladder operators $\Pi_{\pm}$ on the wave functions (107)

$$\psi_{nm}(r,\theta) = g_{nm}\left(\frac{r^2}{a^2}\right)e^{\mathrm{i}m\theta}.$$

This is evaluated as

$$\begin{aligned}
\Pi_{\pm}\psi_{nm}(r,\theta) &= -i\frac{\hbar}{a}\mathrm{e}^{\pm\mathrm{i}\theta}\sqrt{z}\left(2\,\frac{\partial}{\partial z} \pm \mathrm{i}\frac{1}{z}\frac{\partial}{\partial\theta} \pm \kappa\right)\left[g_{nm}(z)e^{\mathrm{i}m\theta}\right] \\
&= -i\frac{\hbar}{a}\mathrm{e}^{\mathrm{i}(m\pm1)\theta}\sqrt{z}\left(2\,\frac{\partial}{\partial z} \mp \frac{m}{z} \pm \kappa\right)g_{nm}(z).
\end{aligned}$$

Let's write g_{nm} (108)

$$g_{nm}(z) = \frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+|m|)!}}\mathrm{e}^{-z/2}z^{|m|/2}L_n^{|m|}(z)$$

in the form

$$g_{nm}(z) = N_{nm}\ell_{nm}(z), \tag{46}$$

where we have set

$$N_{nm} = \frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+|m|)!}} \tag{47}$$

$$\ell_{nm} = \mathrm{e}^{-z/2}z^{|m|/2}L_n^{|m|}(z). \tag{48}$$

We now need to evaluate

$$\Pi_{\pm}\psi_{nm}(r,\theta) = -i\frac{\hbar}{a}N_{nm}\mathrm{e}^{\mathrm{i}(m\pm 1)\theta}\sqrt{z}\left(2\,\frac{\partial}{\partial z} \mp \frac{m}{z} \pm \kappa\right)\ell_{nm}(z),$$

i.e. it is enough if we concentrate on the term

$$\Lambda^{\pm}_{nm} = \sqrt{z}\left(2\,\frac{\partial}{\partial z} \mp \frac{m}{z} \pm \kappa\right)\ell_{nm}(z) = \sqrt{z}\left(2\ell'_{nm} \mp \frac{m}{z}\,\ell_{nm} \pm \kappa\ell_{nm}\right)$$

The derivative of ℓ_{nm} is

$$\ell'_{nm} = \mathrm{e}^{-z/2}z^{|m|/2}\left(-\frac{1}{2}L_n^{|m|}(z) + \frac{|m|}{2z}L_n^{|m|}(z) + L_n^{|m|\prime}(z)\right).$$

We look at the cases of positive and negative angular momenta separately.

1.2.1 Λ^{+}_{nm} and nonnegative angular momenta $m \geq 0$

When $m \geq 0$ our ℓ can be written as

$$\ell_{nm} = \mathrm{e}^{-z/2}z^{m/2}L_n^m(z)$$

and its derivative as

$$\ell'_{nm} = \mathrm{e}^{-z/2}z^{m/2}\left(-\frac{1}{2}L_n^m(z) + \frac{m}{2z}L_n^m(z) + L_n^{m\prime}(z)\right).$$

Using the properties

$$\frac{\mathrm{d}}{\mathrm{d}z}L_n^{\alpha}(z) = -L_{n-1}^{\alpha+1}(z) \qquad (49)$$

$$L_n^{\alpha-1}(z) = L_n^{\alpha}(z) - L_{n-1}^{\alpha}(z) \qquad (50)$$

we now have

$$\begin{aligned}
\Lambda^{+}_{nm} &= \sqrt{z}\left(2\ell'_{nm} - \frac{m}{z}\,\ell_{nm} + \kappa\ell_{nm}\right) \\
&= \mathrm{e}^{-z/2}z^{(m+1)/2}\left(-L_n^m + \frac{m}{z}\,L_n^m + 2L_n^{m\prime} - \frac{m}{z}\,L_n^m + \kappa L_n^m\right) \\
&= \mathrm{e}^{-z/2}z^{(m+1)/2}\left[2\left(L_n^{m\prime} - L_n^m\right) + (1+\kappa)\,L_n^m\right] \\
&= \mathrm{e}^{-z/2}z^{(m+1)/2}\left[2\left(-L_{n-1}^{m+1} - L_n^m\right) + (1+\kappa)\,L_n^m\right] \\
&= \mathrm{e}^{-z/2}z^{(m+1)/2}\left[-2L_n^{m+1} + (1+\kappa)\,L_n^m\right] \\
&= \mathrm{e}^{-z/2}z^{(m+1)/2}\left[-2L_n^{m+1} + (1+\kappa)\left(L_n^{m+1} - L_{n-1}^{m+1}\right)\right] \\
&= \mathrm{e}^{-z/2}z^{(m+1)/2}\left[-\left(1-\kappa\right)L_{n,m+1} - (1+\kappa)\,L_{n-1,m+1}\right] \\
&= -\left(1-\kappa\right)\ell_{n,m+1} - (1+\kappa)\,\ell_{n-1,m+1}.
\end{aligned}$$

To see the effect of Π_+ we need to multiply Λ^+_{nm} by the normalization factor N_{nm} as

$$\begin{aligned}
N_{nm}\Lambda^+_{nm} &= \frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+m)!}}\left[-(1-\kappa)\ell_{n,m+1}-(1+\kappa)\ell_{n-1,m+1}\right] \\
&= -\frac{1}{\sqrt{\pi}a}\Bigg[(1-\kappa)\sqrt{n+m+1}\sqrt{\frac{n!}{(n+m+1)!}}\ell_{n,m+1} \\
&\qquad +(1+\kappa)\sqrt{n}\sqrt{\frac{(n-1)!}{(n-1+m+1)!}}\ell_{n-1,m+1}\Bigg] \\
&= -(1-\kappa)\sqrt{n+m+1}g_{n,m+1}-(1+\kappa)\sqrt{n}g_{n-1,m+1}.
\end{aligned}$$

Hence we have shown that

$$\Pi_+\psi_{nm} = \mathrm{i}\frac{\hbar}{a}\left[(1-\kappa)\sqrt{n+m+1}\psi_{n,m+1}+(1+\kappa)\sqrt{n}\psi_{n-1,m+1}\right].$$

1.2.2 Λ^+_{nm} and negative angular momenta $m<0$

When $m<0$ the function ℓ is

$$\ell_{nm} = \mathrm{e}^{-z/2}z^{-m/2}L_n^{-m}(z)$$

and the derivative

$$\ell'_{nm} = \mathrm{e}^{-z/2}z^{-m/2}\left(-\frac{1}{2}L_n^{-m}(z)-\frac{m}{2z}L_n^{-m}(z)+L_n^{-m\prime}(z)\right).$$

Using, in addition to the property (128) the relations

$$z\frac{\mathrm{d}}{\mathrm{d}z}L_n^\alpha(z) = nL_n^\alpha(z)-(n+\alpha)L^\alpha_{n-1}(z) \qquad (51)$$

$$zL_n^{\alpha+1}(z) = (n+\alpha+1)L_n^\alpha(z)-(n+1)L^\alpha_{n+1}(z) \qquad (52)$$

we now evaluate

$$\begin{aligned}
\Lambda^+_{nm} &= \sqrt{z}\left(2\ell'_{nm}-\frac{m}{z}\ell_{nm}+\kappa\ell_{nm}\right) \\
&= \mathrm{e}^{-z/2}z^{-(m+1)/2}\left(-zL_n^{-m}-mL_n^{-m}+2zL_n^{-m\prime}-mL_n^{-m}+\kappa zL_n^{-m}\right) \\
&= \mathrm{e}^{-z/2}z^{-(m+1)/2}\Big(-zL_n^{-m}-2mL_n^{-m}+2nL_n^{-m}-2(n-m)L^{-m}_{n-1} \\
&\qquad +\kappa zL_n^{-m}\Big) \\
&= \mathrm{e}^{-z/2}z^{-(m+1)/2}\Big[-(1-\kappa)zL_n^{-m}+2n\left(L_n^{-m}-L^{-m}_{n-1}\right) \\
&\qquad -2m\left(L_n^{-m}-L^{-m}_{n-1}\right)\Big] \\
&= \mathrm{e}^{-z/2}z^{-(m+1)/2}\Big\{-(1-\kappa)\left[(n-m)L_n^{-m-1}-(n+1)L^{-m-1}_{n+1}\right]
\end{aligned}$$

$$
\begin{aligned}
& \left.+2(n-m)L_n^{-m-1}\right\} \\
= \quad & \mathrm{e}^{-z/2}z^{-(m+1)/2}\left[(1+\kappa)(n-m)L_n^{-m-1}+(1-\kappa)(n+1)L_{n+1}^{-m-1}\right] \\
= \quad & \mathrm{e}^{-z/2}z^{|m+1|/2}\left[(1+\kappa)(n+|m|)L_n^{|m+1|}+(1-\kappa)(n+1)L_{n+1}^{|m+1|}\right] \\
= \quad & (1+\kappa)(n+|m|)\ell_{n,m+1}+(1-\kappa)(n+1)\ell_{n+1,m+1}.
\end{aligned}
$$

We should note that the relation

$$
-m-1=|m+1|
$$

we used above is not valid when $m=0$.

Recalling that this time we have $|m+1|=-m-1=|m|-1$ we again look at

$$
\begin{aligned}
N_{nm}\Lambda^+_{nm} &= \frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+|m|)!}}\big[(1+\kappa)(n+|m|)\ell_{n,m+1} \\
&\qquad +(1-\kappa)(n+1)\ell_{n+1,m+1}\big] \\
&= \frac{1}{\sqrt{\pi}a}\Bigg[(1+\kappa)\sqrt{n+|m|}\sqrt{\frac{n!}{(n+|m|-1)!}}\ell_{n,m+1} \\
&\qquad +(1-\kappa)\sqrt{n+1}\sqrt{\frac{(n+1)!}{(n+1+|m|-1)!}}\ell_{n+1,m+1}\Bigg] \\
&= \frac{1}{\sqrt{\pi}a}\left[(1+\kappa)\sqrt{n+|m|}g_{n,m+1}+(1-\kappa)\sqrt{n+1}g_{n+1,m+1}\right].
\end{aligned}
$$

Correspondingly we have

$$
\Pi_+\psi_{nm}=-\mathrm{i}\,\frac{\hbar}{a}\left[(1+\kappa)\sqrt{n+|m|}\psi_{n,m+1}+(1-\kappa)\sqrt{n+1}\psi_{n+1,m+1}\right].
$$

1.2.3 Λ^- and positive angular momenta $m>0$

Next we evaluate Λ^-_{nm} recalling that for positive m we have

$$
\begin{aligned}
\ell_{nm} &= \mathrm{e}^{-z/2}z^{m/2}L_n^m(z) \\
\ell'_{nm} &= \mathrm{e}^{-z/2}z^{m/2}\left(-\frac{1}{2}L_n^m(z)+\frac{m}{2z}L_n^m(z)+L_n^{m\prime}(z)\right).
\end{aligned}
$$

We proceed as

$$
\begin{aligned}
\Lambda^-_{nm} &= \sqrt{z}\left(2\ell'_{nm}+\frac{m}{z}\,\ell_{nm}-\kappa\ell_{nm}\right) \\
&= \mathrm{e}^{-z/2}z^{(m-1)/2}\left(-zL_n^m+mL_n^m+2zL_n^{m\prime}+mL_n^m-\kappa\,zL_n^m\right) \\
&= \mathrm{e}^{-z/2}z^{(m-1)/2}\left[-zL_n^m+2mL_n^m+2nL_n^m-2(n+m)L_{n-1}^m-\kappa\,zL_n^m\right] \\
&= \mathrm{e}^{-z/2}z^{(m-1)/2}\big[-zL_n^m+2n\left(L_n^m-L_{n-1}^m\right)
\end{aligned}
$$

$$
\begin{aligned}
& \qquad +2m\left(L_n^m - L_{n-1}^m\right) - \kappa\, z L_n^m\big] \\
= \;& \mathrm{e}^{-z/2} z^{(m-1)/2}\left[-\,(1+\kappa)\, z L_n^m + 2(n+m) L_n^{m-1}\right] \\
= \;& \mathrm{e}^{-z/2} z^{(m-1)/2}\big[2(n+m) L_n^{m-1} \\
& \qquad -(1+\kappa)\left((n+m)L_n^{m-1} - (n+1)L_{n+1}^{m-1}\right)\big] \\
= \;& \mathrm{e}^{-z/2} z^{(m-1)/2}\left[(1-\kappa)(n+m) L_n^{m-1} + (1+\kappa)(n+1) L_{n+1}^{m-1}\right] \\
= \;& (1-\kappa)(n+m)\ell_{n,m-1} + (1+\kappa)(n+1)\ell_{n+1,m-1}.
\end{aligned}
$$

We should again note that this derivation is not valid for $m = 0$ because in that case $m-1$ in L^{m-1} would be negative, contrary to the definition of ℓ.

This time we have $|m-1| = m-1$ and we get

$$
\begin{aligned}
N_{nm}\Lambda_{nm}^- &= \frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+m)!}}\big[(1-\kappa)(n+m)\ell_{n,m-1} \\
&\qquad +(1+\kappa)(n+1)\ell_{n+1,m-1}\big] \\
&= \frac{1}{\sqrt{\pi}a}\Bigg[(1-\kappa)\sqrt{n+m}\sqrt{\frac{n!}{(n+m-1)!}}\ell_{n,m-1} \\
&\qquad +(1+\kappa)\sqrt{n+1}\sqrt{\frac{(n+1)!}{(n+1+m-1)!}}\ell_{n+1,m-1}\Bigg] \\
&= (1-\kappa)\sqrt{n+m}\, g_{n,m-1} + (1+\kappa)\sqrt{n+1}\, g_{n+1,m-1},
\end{aligned}
$$

so that

$$
\Pi_-\psi_{nm} = -\mathrm{i}\,\frac{\hbar}{a}\left[(1-\kappa)\sqrt{n+m}\,\psi_{n,m-1} + (1+\kappa)\sqrt{n+1}\,\psi_{n+1,m-1}\right].
$$

1.2.4 Λ_{nm}^- for nonpositive angular momenta $m \le 0$

For nonpositive m we had

$$
\begin{aligned}
\ell_{nm} &= \mathrm{e}^{-z/2} z^{-m/2} L_n^{-m}(z) \\
\ell_{nm}' &= \mathrm{e}^{-z/2} z^{-m/2}\left(-\frac{1}{2}L_n^{-m}(z) - \frac{m}{2z}L_n^{-m}(z) + L_n^{-m\prime}(z)\right).
\end{aligned}
$$

We evaluate

$$
\begin{aligned}
\Lambda_{nm}^- &= \sqrt{z}\left(2\ell_{nm}' + \frac{m}{z}\,\ell_{nm} - \kappa\ell_{nm}\right) \\
&= \mathrm{e}^{-z/2} z^{(-m+1)/2}\left(-L_n^{-m} - \frac{m}{z}L_n^{-m} + 2L_n^{-m\prime} + \frac{m}{z}L_n^{-m} - \kappa L_n^{-m}\right) \\
&= \mathrm{e}^{-z/2} z^{(-m+1)/2}\left(-(1+\kappa)L_n^{-m} - 2L_{n-1}^{-m+1}\right) \\
&= \mathrm{e}^{-z/2} z^{(-m+1)/2}\left[-(1+\kappa)\left(L_n^{-m+1} - L_{n-1}^{-m+1}\right) - 2L_{n-1}^{-m+1}\right] \\
&= \mathrm{e}^{-z/2} z^{(-m+1)/2}\left(-(1+\kappa)L_n^{-m+1} - (1-\kappa)L_{n-1}^{-m+1}\right) \\
&= \mathrm{e}^{-z/2} z^{|m-1|/2}\left(-(1+\kappa)L_n^{|m-1|} - (1-\kappa)L_{n-1}^{|m-1|}\right) \\
&= -(1+\kappa)\ell_{n,m-1} - (1-\kappa)\ell_{n-1,m-1}.
\end{aligned}
$$

This is valid also in the special case $m = 0$ because even then we have $1 - m = |1 - m|$.

Because this time we have $|m-1| = 1 - m = 1 + |m|$ we can write

$$
\begin{aligned}
N_{nm}\Lambda^{-}_{nm} &= -\frac{1}{\sqrt{\pi}a}\sqrt{\frac{n!}{(n+|m|)!}}\left[(1+\kappa)\ell_{n,m-1} + (1-\kappa)\ell_{n-1,m-1}\right] \\
&= -\frac{1}{\sqrt{\pi}a}\left[(1+\kappa)\sqrt{n+|m|+1}\sqrt{\frac{n!}{(n+|m|+1)!}}\ell_{n,m-1}\right. \\
&\qquad \left. +(1-\kappa)\sqrt{n}\sqrt{\frac{(n-1)!}{(n-1+|m|+1)!}}\ell_{n-1,m-1}\right] \\
&= -(1+\kappa)\sqrt{n+|m-1|}g_{n,m-1} - (1-\kappa)\sqrt{n}g_{n-1,m-1}
\end{aligned}
$$

and, correspondingly

$$
\Pi_{-}\psi_{nm} = \mathrm{i}\,\frac{\hbar}{a}\left[(1+\kappa)\sqrt{n+|m-1|}\psi_{n,m-1} + (1-\kappa)\sqrt{n}\psi_{n-1,m-1}\right].
$$

1.2.5 Matrix elements

We define the state vectors $|nm\rangle$ via the relation

$$
\psi_{nm}(\vec{r}) = \langle\vec{r}|nm\rangle \tag{53}
$$

and collect our results as

$$
\begin{aligned}
\Pi_{+}|nm\rangle &= \mathrm{i}\,\frac{\hbar}{a}\big[(1-\kappa)\sqrt{n+m+1}|n,m+1\rangle \\
&\qquad +(1+\kappa)\sqrt{n}|n-1,m+1\rangle\big]; \quad m \geq 0 \qquad (54) \\
\Pi_{+}|nm\rangle &= -\mathrm{i}\,\frac{\hbar}{a}\big[(1+\kappa)\sqrt{n+|m|}|n,m+1\rangle \\
&\qquad +(1-\kappa)\sqrt{n+1}|n+1,m+1\rangle\big]; \quad m < 0 \qquad (55) \\
\Pi_{-}|nm\rangle &= -\mathrm{i}\,\frac{\hbar}{a}\big[(1-\kappa)\sqrt{n+m}|n,m-1\rangle \\
&\qquad +(1+\kappa)\sqrt{n+1}|n+1,m-1\rangle\big]; \quad m > 0 \qquad (56) \\
\Pi_{-}|nm\rangle &= \mathrm{i}\,\frac{\hbar}{a}\big[(1+\kappa)\sqrt{n+|m-1|}|n,m-1\rangle \\
&\qquad +(1-\kappa)\sqrt{n}|n-1,m-1\rangle\big]; \quad m \leq 0. \qquad (57)
\end{aligned}
$$

Because the SO Hamiltonian H_{SO} (118) was

$$
H_{\mathrm{SO}} = \mathrm{i}\,\frac{\alpha}{\hbar}\begin{pmatrix} 0 & \Pi_{-} \\ -\Pi_{+} & 0 \end{pmatrix},
$$

and because Π_{-} decreases and Π_{+} increases the orbital angular momentum by one whereas H^0_{s} is diagonal in the spin space and conserves the orbital angular momentum the eigenstates of the single particle Hamiltonian

$$H_{\rm SP} = H_{\rm S}^0 + H_{\rm SO}$$

must be of the form

$$\eta = \begin{pmatrix} g_m^{\uparrow} \\ g_{m+1}^{\downarrow} \end{pmatrix}, \tag{58}$$

where $g_m^{\uparrow,\downarrow}$ are eigenstates of the orbital angular momentum with eigenvalue m. In particular, the matrix elemnts of $H_{\rm SO}$ between the basic spinor states are

$$\begin{aligned}
{\rm i}\frac{\alpha}{\hbar}\begin{pmatrix} |n'm'\rangle \\ 0 \end{pmatrix}^{\dagger}\begin{pmatrix} 0 & \Pi_- \\ -\Pi_+ & 0 \end{pmatrix}\begin{pmatrix} |nm\rangle \\ 0 \end{pmatrix} &= 0 \\
{\rm i}\frac{\alpha}{\hbar}\begin{pmatrix} 0 \\ |n'm'\rangle \end{pmatrix}^{\dagger}\begin{pmatrix} 0 & \Pi_- \\ -\Pi_+ & 0 \end{pmatrix}\begin{pmatrix} 0 \\ |nm\rangle \end{pmatrix} &= 0
\end{aligned}$$

and

$$\begin{aligned}
{\rm i}\frac{\alpha}{\hbar}\begin{pmatrix} 0 \\ |n'm'\rangle \end{pmatrix}^{\dagger}\begin{pmatrix} 0 & \Pi_- \\ -\Pi_+ & 0 \end{pmatrix}\begin{pmatrix} |nm\rangle \\ 0 \end{pmatrix} &= -{\rm i}\frac{\alpha}{\hbar}\langle n'm'|\Pi_+|nm\rangle \\
{\rm i}\frac{\alpha}{\hbar}\begin{pmatrix} |n'm'\rangle \\ 0 \end{pmatrix}^{\dagger}\begin{pmatrix} 0 & \Pi_- \\ -\Pi_+ & 0 \end{pmatrix}\begin{pmatrix} 0 \\ |nm\rangle \end{pmatrix} &= {\rm i}\frac{\alpha}{\hbar}\langle n'm'|\Pi_-|nm\rangle .
\end{aligned}$$

The latter equations evaluate further to

$$\begin{aligned}
&\begin{pmatrix} 0 \\ |n'm'\rangle \end{pmatrix}^{\dagger} H_{\rm SO}\begin{pmatrix} |nm\rangle \\ 0 \end{pmatrix} \\
&\quad = \frac{\alpha}{a}\Big[(1-\kappa)\sqrt{n+m+1}\langle n'm'|n,m+1\rangle \\
&\qquad +(1+\kappa)\sqrt{n}\langle n'm'|n-1,m+1\rangle\Big]\,,\ m\geq 0 \\
&\quad = -\frac{\alpha}{a}\Big[(1+\kappa)\sqrt{n+|m|}\langle n'm'|n,m+1\rangle \\
&\qquad +(1-\kappa)\sqrt{n+1}\langle n'm'|n+1,m+1\rangle\Big]\,,\ m<0 \\
&\begin{pmatrix} |n'm'\rangle \\ 0 \end{pmatrix}^{\dagger} H_{\rm SO}\begin{pmatrix} 0 \\ |nm\rangle \end{pmatrix} \\
&\quad = \frac{\alpha}{a}\Big[(1-\kappa)\sqrt{n+m}\langle n'm'|n,m-1\rangle \\
&\qquad +(1+\kappa)\sqrt{n+1}\langle n'm'|n+1,m-1\rangle\Big]\,,\ m>0 \\
&\quad = -\frac{\alpha}{a}\Big[(1+\kappa)\sqrt{n+|m-1|}\langle n'm'|n,m-1\rangle \\
&\qquad +(1-\kappa)\sqrt{n}\langle n'm'|n-1,m-1\rangle\Big]\,,\ m\leq 0.
\end{aligned}$$

We expand the functions $g_m^{\uparrow,\downarrow}$ as

$$g_m^{\sigma} = \sum_n c_{nm}^{\sigma}|nm\rangle. \tag{59}$$

The two component spinors can be correspondingly expanded as

$$\eta = \sum_n c^{\uparrow}_{nm} \begin{pmatrix} |nm\rangle \\ 0 \end{pmatrix} + \sum_n c^{\downarrow}_{nm} \begin{pmatrix} 0 \\ |n, m+1\rangle \end{pmatrix}. \tag{60}$$

Now, the effect of $H_{\rm SO}$ on η is

$$\begin{aligned} H_{\rm SO}\eta &= {\rm i}\,\frac{\alpha}{\hbar} \begin{pmatrix} 0 & \Pi_- \\ -\Pi_+ & 0 \end{pmatrix} \Bigg[\sum_n c^{\uparrow}_{nm} \begin{pmatrix} |nm\rangle \\ 0 \end{pmatrix} \\ &\qquad\qquad + \sum_n c^{\downarrow}_{nm} \begin{pmatrix} 0 \\ |n, m+1\rangle \end{pmatrix} \Bigg] \\ &= {\rm i}\,\frac{\alpha}{\hbar} \Bigg[\sum_n c^{\uparrow}_{nm} \begin{pmatrix} 0 \\ -\Pi_+|nm\rangle \end{pmatrix} + \sum_n c^{\downarrow}_{nm} \begin{pmatrix} \Pi_-|n, m+1\rangle \\ 0 \end{pmatrix} \Bigg], \end{aligned}$$

so we clearly need to consider only the matrix elements $\langle n', m+1|\Pi_+|nm\rangle$ and $\langle n'm|\Pi_-|n, m+1\rangle$. The former one gives

$$\begin{aligned} &\langle n', m+1|\Pi_+|nm\rangle = \\ &\quad {\rm i}\,\frac{\hbar}{a} \begin{cases} (1-\kappa)\sqrt{n+m+1}\,\delta_{n'n} + (1+\kappa)\sqrt{n}\,\delta_{n',n-1}; & m \geq 0 \\ -(1+\kappa)\sqrt{n+|m|}\,\delta_{n'n} - (1-\kappa)\sqrt{n+1}\,\delta_{n',n+1}; & m < 0 \end{cases} \end{aligned} \tag{61}$$

and the latter one

$$\begin{aligned} &\langle n', m|\Pi_-|n, m+1\rangle = \\ &\quad -{\rm i}\,\frac{\hbar}{a} \begin{cases} (1-\kappa)\sqrt{n+m+1}\,\delta_{n'n} + (1+\kappa)\sqrt{n+1}\,\delta_{n',n+1}; & m > 0 \\ -(1+\kappa)\sqrt{n+|m|}\,\delta_{n'n} - (1-\kappa)\sqrt{n}\,\delta_{n',n-1}; & m \leq 0. \end{cases} \end{aligned} \tag{62}$$

From now on we assume that the orbital angular momentum of the upper components of the spinors are fixed to m and the lower ones to $m+1$. Defining the spinors

$$|n;\uparrow\rangle = \begin{pmatrix} |n, m\rangle \\ 0 \end{pmatrix} \tag{63}$$

$$|n;\downarrow\rangle = \begin{pmatrix} 0 \\ |n, m+1\rangle \end{pmatrix} \tag{64}$$

and the short hand

$$\eta^{\pm} = 1 \pm \kappa \tag{65}$$

we see that

$$\begin{aligned} \langle n';\uparrow|H_{\rm SO}|n;\uparrow\rangle &= 0 \\ \langle n';\uparrow|H_{\rm SO}|n;\downarrow\rangle &= \frac{\alpha}{a} \begin{cases} \eta^-\sqrt{n+m+1}\,\delta_{n'n} + \eta^+\sqrt{n+1}\,\delta_{n',n+1}; & m > 0 \\ -\eta^+\sqrt{n+|m|}\,\delta_{n'n} - \eta^-\sqrt{n}\,\delta_{n',n-1}; & m \leq 0 \end{cases} \\ \langle n';\downarrow|H_{\rm SO}|n;\uparrow\rangle &= \frac{\alpha}{a} \begin{cases} \eta^-\sqrt{n+m+1}\,\delta_{n'n} + \eta^+\sqrt{n}\,\delta_{n',n-1}; & m \geq 0 \\ -\eta^+\sqrt{n+|m|}\,\delta_{n'n} - \eta^-\sqrt{n+1}\,\delta_{n',n+1}; & m < 0 \end{cases} \\ \langle n';\downarrow|H_{\rm SO}|n;\downarrow\rangle &= 0. \end{aligned}$$

We now consider positive and negative orbital angular momenta separately. For angular momenta $m \geq 0$ we define

$$
\begin{aligned}
|2n;+\rangle &= |n;\uparrow\rangle = \begin{pmatrix} |nm\rangle \\ 0 \end{pmatrix} && (66)\\
z_{2n}^{+} &= c_{nm}^{\uparrow} && (67)\\
|2n+1;+\rangle &= |n;\downarrow\rangle = \begin{pmatrix} 0 \\ |n,m+1\rangle \end{pmatrix} && (68)\\
z_{2n+1}^{+} &= c_{nm}^{\downarrow} && (69)
\end{aligned}
$$

and for the angular momenta $m < 0$

$$
\begin{aligned}
|2n;-\rangle &= |n;\downarrow\rangle = \begin{pmatrix} 0 \\ |n,m+1\rangle \end{pmatrix} && (70)\\
z_{2n}^{-} &= c_{nm}^{\downarrow} && (71)\\
|2n+1;-\rangle &= |n;\uparrow\rangle = \begin{pmatrix} |n,m\rangle \\ 0 \end{pmatrix} && (72)\\
z_{2n+1}^{-} &= c_{nm}^{\uparrow}. && (73)
\end{aligned}
$$

The matrix elements of H_{SO} now read

$$
\begin{aligned}
\langle 2n-1;+|H_{\mathrm{SO}}|2n;+\rangle &= \langle n-1;\downarrow|H_{\mathrm{SO}}|n;\uparrow\rangle = \eta^{+}\frac{\alpha}{a}\sqrt{n}\\
\langle 2n;+|H_{\mathrm{SO}}|2n;+\rangle &= \langle n;\uparrow|H_{\mathrm{SO}}|n;\uparrow\rangle = 0\\
\langle 2n+1;+|H_{\mathrm{SO}}|2n;+\rangle &= \langle n;\downarrow|H_{\mathrm{SO}}|n;\uparrow\rangle = \eta^{-}\frac{\alpha}{a}\sqrt{n+m+1}\\
\langle 2n;+|H_{\mathrm{SO}}|2n+1;+\rangle &= \langle n;\uparrow|H_{\mathrm{SO}}|n;\downarrow\rangle = \eta^{-}\frac{\alpha}{a}\sqrt{n+m+1}\\
\langle 2n+1;+|H_{\mathrm{SO}}|2n+1;+\rangle &= \langle n;\downarrow|H_{\mathrm{SO}}|n;\downarrow\rangle = 0\\
\langle 2n+2;+|H_{\mathrm{SO}}|2n+1;+\rangle &= \langle n+1;\uparrow|H_{\mathrm{SO}}|n;\downarrow\rangle = \eta^{+}\frac{\alpha}{a}\sqrt{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle 2n-1;-|H_{\mathrm{SO}}|2n;-\rangle &= \langle n-1;\uparrow|H_{\mathrm{SO}}|n;\downarrow\rangle = -\eta^{-}\frac{\alpha}{a}\sqrt{n}\\
\langle 2n;-|H_{\mathrm{SO}}|2n;-\rangle &= \langle n;\downarrow|H_{\mathrm{SO}}|n;\downarrow\rangle = 0\\
\langle 2n+1;-|H_{\mathrm{SO}}|2n;-\rangle &= \langle n;\uparrow|H_{\mathrm{SO}}|n;\downarrow\rangle = -\eta^{+}\frac{\alpha}{a}\sqrt{n+|m|}\\
\langle 2n;-|H_{\mathrm{SO}}|2n+1;-\rangle &= \langle n;\downarrow|H_{\mathrm{SO}}|n;\uparrow\rangle = -\eta^{+}\frac{\alpha}{a}\sqrt{n+|m|}\\
\langle 2n+1;-|H_{\mathrm{SO}}|2n+1;-\rangle &= \langle n;\uparrow|H_{\mathrm{SO}}|n;\uparrow\rangle = 0\\
\langle 2n+2;-|H_{\mathrm{SO}}|2n+1;-\rangle &= \langle n+1;\downarrow|H_{\mathrm{SO}}|n;\uparrow\rangle = -\eta^{-}\frac{\alpha}{a}\sqrt{n+1}.
\end{aligned}
$$

We now check that the matrix with matrix elements $\langle i;\pm|H_{\mathrm{SO}}|j;\pm\rangle$ is Hermitian or actually, because the matrix elements are real, symmetric. First, for

$m \geq 0$ we have

$$
\begin{aligned}
\langle i;+|H_{\rm SO}|j;+\rangle &= \frac{\alpha}{a}\begin{cases} \eta^+\sqrt{n}; & i=2n-1, \quad j=2n \\ \eta^-\sqrt{n+m+1}; & i=2n+1, \quad j=2n \\ \eta^-\sqrt{n+m+1}; & i=2n, \qquad j=2n+1 \\ \eta^+\sqrt{n+1}; & i=2n+2, \quad j=2n+1 \end{cases} \\
&= \frac{\alpha}{a}\begin{cases} \eta^+\sqrt{n}; & i=2n-1, \quad j=2n \\ \eta^-\sqrt{n+m+1}; & i=2n+1, \quad j=2n \\ \eta^-\sqrt{n+m+1}; & i=2n, \qquad j=2n+1 \\ \eta^+\sqrt{n}; & i=2n, \qquad j=2n-1 \end{cases} \\
&= \frac{\alpha}{a}\begin{cases} \eta^+\sqrt{n}; & i=2n, \qquad j=2n-1 \\ \eta^-\sqrt{n+m+1}; & i=2n, \qquad j=2n+1 \\ \eta^-\sqrt{n+m+1}; & i=2n+1, \quad j=2n \\ \eta^+\sqrt{n}; & i=2n-1, \quad j=2n \end{cases} \\
&= \langle j;+|H_{\rm SO}|i;+\rangle
\end{aligned}
$$

and next, for $m < 0$

$$
\begin{aligned}
\langle i;-|H_{\rm SO}|j;-\rangle &= -\frac{\alpha}{a}\begin{cases} \eta^-\sqrt{n}; & i=2n-1, \quad j=2n \\ \eta^+\sqrt{n+|m|}; & i=2n+1, \quad j=2n \\ \eta^+\sqrt{n+|m|}; & i=2n, \qquad j=2n+1 \\ \eta^-\sqrt{n+1}; & i=2n+2, \quad j=2n+1 \end{cases} \\
&= -\frac{\alpha}{a}\begin{cases} \eta^-\sqrt{n}; & i=2n-1, \quad j=2n \\ \eta^+\sqrt{n+|m|}; & i=2n+1, \quad j=2n \\ \eta^+\sqrt{n+|m|}; & i=2n, \qquad j=2n+1 \\ \eta^-\sqrt{n}; & i=2n, \qquad j=2n-1 \end{cases} \\
&= -\frac{\alpha}{a}\begin{cases} \eta^-\sqrt{n}; & i=2n, \qquad j=2n-1 \\ \eta^+\sqrt{n+|m|}; & i=2n, \qquad j=2n+1 \\ \eta^+\sqrt{n+|m|}; & i=2n+1, \quad j=2n \\ \eta^-\sqrt{n}; & i=2n-1, \quad j=2n \end{cases} \\
&= \langle j;-|H_{\rm SO}|i;-\rangle.
\end{aligned}
$$

We can see that in the matrix representation $H_{\rm SO}$ is a symmetric tridiagonal matrix with vanishing diagonal and with subdiagonals

$$
\langle i+1;+|H_{\rm SO}|i;+\rangle = \frac{\alpha}{a}\begin{cases} \eta^-\sqrt{n+m+1}; & i=2n+1 \\ \eta^+\sqrt{n+1}; & i=2n+2 \end{cases} \tag{74}
$$

and

$$
\langle i+1;-|H_{\rm SO}|i;-\rangle = -\frac{\alpha}{a}\begin{cases} \eta^+\sqrt{n+|m|}; & i=2n+1 \\ \eta^-\sqrt{n+1}; & i=2n+2. \end{cases} \tag{75}
$$