Quantum Field Theory Final exam, 10.9. 2008

Answering correctly to 4 questions gives full points; however, answering to all 5 contributes to the total point score.

1. Let us consider (non-relativistic) harmonic oscillator with

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\,,$$

   and define operators

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\hat{x} + \frac{i}{\sqrt{2m\omega}}\hat{p} \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\hat{x} - \frac{i}{\sqrt{2m\omega}}\hat{p}.$$

   Using

$$[\hat{x},\hat{p}] = i, \qquad [\hat{x},\hat{x}] = [\hat{p},\hat{p}] = 0$$

   show that

   a) $[\hat{a},\hat{a}] = [\hat{a}^\dagger,\hat{a}^\dagger] = 0$

   b) $[\hat{a},\hat{a}^\dagger] = 1$,

   c) $\hat{H} = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

2. Let $c = (c_1 \;\; c_2)^T$, $c^\dagger = (c_1^* \;\; c_2^*)$ be 2-component Grassmann vectors, and M a 2×2 complex matrix. Calculate the Grassmann integral

$$\int \mathrm{d}c_1^*\,\mathrm{d}c_1\,\mathrm{d}c_2^*\,\mathrm{d}c_2\;\; \exp[-c^\dagger M c]$$

3. Show that the solution of the non-homogeneous equation ($J(x)$: source term)

$$(\partial^\mu\partial_\mu + m^2)\psi(x) = J(x)$$

   can be written as

$$\psi(x) = \psi_0(x) + i\int d^4y G(x,y)J(y)$$

   where ψ_0 is a solution for the homogeneous equation and G is a Green function. Convince yourself that if G is the retarded propagator, $J(y)$ can affect $\psi(x)$ only if y is in the past light cone of point x (i.e. $y^0 \le x^0$, $(x-y)^2 \ge 0$).

4. Let us consider the "scalar quantum electrodynamics", a theory with a complex scalar field $\phi(x)$ and an abelian gauge field $A_\mu(x)$. The action is

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + (D_\mu(x)\phi(x))^*\, D^\mu(x)\phi(x) - m^2\phi^*(x)\phi(x)\right], \qquad (1)$$

   where

$$D_\mu(x) = \partial_\mu - ieA_\mu(x), \qquad F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),$$

   and e is the coupling constant.
   Show that the action is invariant under the gauge transformation ($\theta(x)$ a real number)

$$\phi(x) \to e^{i\theta(x)}\phi(x),\;\; \phi^*(x) \to e^{-i\theta(x)}\phi^*(x),\;\; A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x).$$

5. In scalar quantum electrodynamics (eq.(1)),

a) write down the Feynman propagator $G(p)$ for the scalar field ϕ.

b) For the free gauge field, show that we can write

$$\int d^4x - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) = \int d^4x\, d^4y \frac{1}{2} A_\mu(x) [g^{\mu\nu} \partial_\alpha \partial^\alpha - \partial^\mu \partial^\nu] \delta^4(x-y) A_\nu(y).$$

c) Using the Lorenz gauge $\partial_\nu A^\nu = 0$, argue that the gauge field Feynman propagator is

$$D_{\mu\nu}(q) = \frac{-i g_{\mu\nu}}{q^2 + i\epsilon}.$$

(Note that in this case you need not to use the gauge parameter ξ, but can impose the gauge condition directly. Ignore ghosts.)