Introduction to quantum field theory

Homework 1, Friday 17.10.2003

1. Show that the volume of n-dimensional radius R sphere is
$$V = \frac{R^n}{n}\frac{2\pi^{n/2}}{\Gamma(n/2)}.$$
(We shall need this later)

2. Consider the Lagrange density of a N component scalar field:
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - V(\phi_1,\phi_2,\ldots,\phi_N)$$
(implicit summation over repeated index). Assuming that the action is invariant under global transformations
$$\delta\phi_k = i\varepsilon\lambda_{k\ell}\phi_\ell,$$
find the conserved Noether-current and charge. What is the form of $\lambda_{k\ell}$, if $V = V(\phi^a\phi^a)$?

3. The action of the free complex scalar field ϕ is invariant under global U(1)-transformations $\phi \to e^{i\theta}\phi$. Show that the conserved charge is
$$Q = i\int d^3\vec{x}(\phi^*\dot{\phi} - \phi\dot{\phi}^*)$$
and that Q is conserved also in quantum theory, i.e. $[H,Q] = 0$. What is $Q\phi^\dagger(\vec{x},t)|0\rangle$?

4. The commutation relation for the generators of Lorentz-transformations is
$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho})$$
Let us define the generators for rotations and boosts:
$$L^i = \frac{1}{2}\epsilon^{ijk}M^{jk} \qquad K^i = M^{0i}.$$
Infinitesimal Lorentz-transformation can be written as
$$\Phi \to (1 - i\vec{\theta}\cdot\vec{L} - i\vec{\beta}\cdot\vec{K})\Phi$$
Write out the commutation relations for the vector operators $\vec{L}, \vec{K}$. Show that the combinations
$$\vec{J}_+ = \frac{1}{2}(\vec{L} + i\vec{K}), \quad \vec{J}_- = \frac{1}{2}(\vec{L} - i\vec{K})$$
commute with each other and separately satisfy the commutation relations for the angular momentum operators.

   This implies that the representations of the Lorentz-group are equivalent to representations of L, so that there are 2 independent quantum numbers: j_+ and j_-. The reps are characterized by the pair (j_+, j_-). $(0,0)$ is scalar, $(\frac{1}{2},0)$ left-handed spinor, $(0,\frac{1}{2})$ right-handed spinor, $(\frac{1}{2},\frac{1}{2})$ vector.

5. Consider the theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3}\lambda\phi^3.$$

where ϕ is a scalar field. What is the Feynman propagator? Write 2,3 and 4-point functions $\langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle$, $n = 2, 3, 4$, to the order λ^2 (as was done on page 47 for $\lambda\phi^4$ theory). What about order λ^3?