On page 49 there is an error, let us do it here more precisely:

$$\langle \varphi_b | e^{-i\hat{H}t} | \varphi_a \rangle = \int \prod_{i=1}^{N-1} d\varphi_i \, \langle \varphi_a | e^{-i\hat{H}\delta t} | \varphi_{N-1} \rangle \times \ldots \langle \varphi_1 | e^{-i\hat{H}\delta t} | \varphi_a \rangle$$

- Consider one matrix element:

$$\langle \varphi_{i+1} | e^{-i\hat{H}\delta t} | \varphi_i \rangle = \int \left[\frac{d\pi_i}{2\pi} \right] \langle \varphi_{i+1} | \pi_i \rangle \langle \pi_i | e^{-i\hat{H}\delta t} | \varphi_i \rangle$$

- For any reasonable $\hat{H} = H(\hat{\varphi}, \hat{\pi})$ we have

$$\langle \pi_i | e^{-i\hat{H}\delta t} | \varphi_i \rangle = e^{-iH(\varphi_i, \pi_i)\delta t} \langle \pi_i | \varphi_i \rangle + O(\delta t^2)$$

More concretely, we specify (cf. page 16)

$$H = \int d^3\bar{x} \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\bar{\nabla}\varphi)^2 + V(\varphi) \right] \tag{2.16b}$$

- Now the matrix element $\left(\langle \varphi | \pi \rangle = e^{i\int d^3\bar{x}\, \varphi\pi} \right)$

$$\langle \varphi_{i+1} | e^{-i\hat{H}\delta t} | \varphi_i \rangle = \int \left[\prod_{\bar{x}} \frac{d\pi_i(\bar{x})}{(2\pi)} \right] e^{i\int d^3\bar{x}\, \pi_i \underbrace{(\varphi_{i+1} - \varphi_i)}_{\simeq \dot{\varphi}_i \cdot \delta t} - iH\delta t}$$

$$= \int \left[\prod_{\bar{x}} \frac{d\pi_i(\bar{x})}{2\pi} \right] \exp\left[i\int d^3\bar{x} \left(\pi_i \dot{\varphi}_i - \tfrac{1}{2}\pi_i^2 - \tfrac{1}{2}(\bar{\nabla}\varphi)^2 - V(\varphi) \right) \delta t \right] \tag{2.16c}$$

Integral over π_i is gaussian: thus,

$$\tfrac{1}{2}\pi_i^2 - \pi_i \dot{\varphi}_i = \tfrac{1}{2}(\pi_i - \dot{\varphi}_i)^2 - \tfrac{1}{2}\dot{\varphi}_i^2$$

Change $\pi_i \to \pi_i' = \pi_i - \dot{\varphi}_i$

and we obtain

$$\langle \varphi_{i+1} | e^{-i\hat{H}\delta t} | \varphi_i \rangle = \left[\prod_{\bar{x}} \frac{1}{\sqrt{2\pi i}} \right] \times e^{i\delta t \int d^3\bar{x} \left[\frac{1}{2} \partial_\nu \varphi \partial^\nu \varphi - V(\varphi) \right]} \tag{2.16d}$$

(funny constant: $(2\pi i)^{-N/2}$, where N = number of coordinate points ! (?))

$$= C\, e^{i\delta t \int d^3\bar{x}\, \mathcal{L}} \tag{2.16e}$$

Thus,

$$\langle \varphi_b | e^{-i\hat{H}t} | \varphi_a \rangle = \int \prod_{i=1}^{N-1} \left[\prod_{\bar{x}} d\varphi_i(\bar{x}) \right] e^{i \sum_{i=1}^{N} \delta t \int d^3\bar{x}\, \mathcal{L}(\varphi_i)}$$

$$= \underline{\int \left[\prod_{x} d\varphi(x) \right] e^{i \int d^4x\, \mathcal{L}(\varphi)}} \tag{2.16f}$$

where we do not worry about the constant in front $= (2\pi i)^{-N/2}$, where N is now # of space-<u>time</u> points.

In the last step we used $\varphi(x) = \varphi(\delta t \cdot i, \bar{x}) \equiv \varphi_i(x)$

and boundary conditions are

$$\varphi(0, \bar{x}) = \varphi_a(\bar{x}) \; ; \quad \varphi(t, \bar{x}) = \varphi_b(\bar{x})$$