Note: $\left.\frac{\partial}{\partial \chi_i}\right|_{\chi=0} e^{\theta_j^* \chi_j} = -\theta_i^*$; $\left.\frac{\partial}{\partial \chi_i^*}\right|_{\chi^*=0} e^{\chi_j^* \theta_j} = \theta_i$

Thus, we obtain

$$\left.\frac{\partial}{\partial \chi_i^*}\frac{\partial}{\partial \chi_j} Z[\chi,\chi^*]\right|_{\chi,\chi^*=0} = \int [\Pi d\theta^* d\theta]\, \theta_i \theta_j^* e^{-\theta^\dagger M \theta} \tag{4.40}$$

$$\equiv \langle \theta_i \theta_j^* \rangle = -\langle \theta_j^* \theta_i \rangle = -\det M \, (M^{-1})_{ij}$$

Note: $$\langle \theta_i \theta_j \rangle = \langle \theta_i^* \theta_j^* \rangle = 0 \tag{4.41}$$

$$\langle \theta_i \rangle = \langle \theta_i^* \rangle = 0$$

$$\langle \theta_i \theta_j \theta_k^* \theta_l^* \rangle = \wick{\c1\theta_i \c2\theta_j \c2\theta_k^* \c1\theta_l^*} - \wick{\c1\theta_i \c2\theta_j \c1\theta_k^* \c2\theta_l^*} \tag{4.42}$$

where $\wick{\c\theta_i \c\theta_j^*} \equiv \langle \theta_i \theta_j^* \rangle$

(compare with free scalar, (2.28))

In general,

$$\langle \theta_1 \theta_2 .. \theta_N \theta_1^* \theta_2^* .. \theta_N^* \rangle = \sum (-1)^n \text{(pairwise contractions)}$$

$n = \#$ crossed lines

Path integral for free fermions

We can now write the "classical" fermion action as

$$S[\psi,\bar{\psi}] = \int d^4x\, \bar{\psi}(i\not{\partial} - m)\psi \qquad (4.43)$$

where $\bar{\psi} = \psi^{\dagger}\gamma^0$ and ψ are 4-comp. Grassmann spinors, taken to be independent. Path integral is defined with Grassmann sources $\eta, \bar{\eta}$:

$$\boxed{Z[\eta,\bar{\eta}] = Z[0]^{-1} \int D\psi D\bar{\psi}\, e^{iS[\psi,\bar{\psi}] + i\int d^4x(\bar{\eta}\psi + \bar{\psi}\eta)}} \qquad (4.44)$$

Why Grassmann? $\psi, \bar{\psi}$ have to anticommute in order to get correct anticommutation relations, (4.28). These, in turn, are needed to guarantee positive energy to antiparticles (not shown here)

Completing the squares:

$$\bar{\psi} \to \bar{\psi} - \bar{\eta}(i\not{\partial}-m)^{-1}, \quad \psi \to \psi - (i\not{\partial}-m)^{-1}\eta$$

$$Z[\eta,\bar{\eta}] = \exp\left[-\int d^4x\, d^4y\, \bar{\eta}(x)\, S_F(x-y)\, \eta(y)\right] \qquad (4.45)$$

where Feynman propagator for fermions is

$$S_F(x-y) = \text{"}\frac{i}{i\not{\partial} - m}\text{"} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p} - m + i\epsilon} e^{-ip\cdot(x-y)} \qquad (4.46)$$

Where still

$$\underline{\frac{i}{\not p - m + i\epsilon} = i\,\frac{\not p + m}{p^2 - m^2 + i\epsilon}} \qquad (4.47)$$

Thus, S_F is the Green function for Dirac:

$$(i\not\partial_x - m)\, S_F(x-y) = i\,\delta^{(4)}(x-y) \qquad (4.48)$$

Thus,

$$\underset{\uparrow\,\text{4x4-matrix!}}{\langle \psi(x)\,\bar\psi(y)\rangle} = \frac{\overset{\swarrow\,\text{(-) and } i^2 \text{ cancel!}}{-}\,\partial}{i\,\partial\bar\eta(x)}\,\frac{\partial}{i\,\partial\eta(y)}\, Z[\eta,\bar\eta]\Big|_{\eta=0}$$

$$= \underline{S_F(x-y)} \qquad (4.49)$$

$$\langle \tilde\psi(p)\,\tilde{\bar\psi}(q)\rangle = \delta^{(4)}(p-q)\; i\,\frac{\not p + m}{p^2 - m^2 + i\epsilon} \qquad (4.50)$$

These are analogous to free boson propagators!

NOTE: These generalize to <u>euclidean</u> formalism and to <u>$d = 4 - 2\epsilon$</u> dimensions just like the scalar propagators.

4.4. Symmetries

1. Chiral symmetry

If $m=0$, and writing $\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$

ψ_L and ψ_R do not mix in action (4.43).

Defining

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}-\mathbb{1} & 0 \\ 0 & \mathbb{1}\end{pmatrix} \qquad (4.51)$$

↖ Weyl rep.

we note $\{\gamma^5, \gamma^\mu\} = 0$, $(\gamma^5)^2 = \mathbb{1}$

$\gamma^{5\dagger} = \gamma^5$ $\quad (\gamma^{0\dagger} = \gamma^0,\ \gamma^{i\dagger} = -\gamma^i)$

We see that (4.43) is invariant under transformations

A) $\psi \to e^{i\alpha}\psi \quad , \quad \bar{\psi} \to \bar{\psi} e^{-i\alpha}$ (4.52)

and

B) $\psi \to e^{i\beta\gamma^5}\psi \quad , \quad \bar{\psi} \to \bar{\psi} e^{i\beta\gamma^5}$ (4.53)

A) is valid for any m, and the Noether current is (check!)

$$\underline{j^\mu = \bar{\psi}\gamma^5\psi} \qquad (4.54)$$

corresponding to conserved (electric) charge.
(or conserved particle-antiparticle number)

B) is symmetry only if $m=0$. It is called chiral symmetry, with the corresponding conserved Noether current, axial vector current

$$j^{5\nu} = \bar{\psi}\gamma^{\nu}\gamma^{5}\psi$$

The projection operators $P_L = \frac{1}{2}(1-\gamma^5)$ and $P_R = \frac{1}{2}(1+\gamma^5)$ project ψ_L and ψ_R out of ψ.

Chiral symmetry turns out to have a central role in the structure of the standard model (not discussed here)

2. Discrete symmetries: C, P, T

- Above symmetries were continuous, i.e. there is a continuous parameter (α, β) which describes the transformation. When parameter $= 0$, transformation = identity. (Lorentz-transformations too!)

- In addition, fermion lagrangian in (4.43) also has several <u>discrete</u> symmetries

Symmetries under

T: time reversal $x_0 \to -x_0$

P: parity $\bar{x} \to -\bar{x}$

C: charge conjugation particle ⟷ antiparticle

have a central role

Note: T and P are part of (enlarged) Lorentz transformations, <u>preserve $a^2 = a_\mu a^\nu$</u>

$$\begin{tikzcd}
\text{Std. Lorentz} \arrow[r] & L_+^\uparrow \arrow[r, leftrightarrow, "P"] \arrow[d, leftrightarrow, "T"] & L_-^\uparrow = PL_+^\uparrow \arrow[d, leftrightarrow, "T"'] & \text{"orthocronous"} \\
\text{proper, orthogonous} & L_+^\downarrow = TL_+^\uparrow \arrow[r, leftrightarrow, "P"] & L_-^\downarrow = PTL_+^\uparrow & \text{"non-orthogonous"} \\
 & \text{"proper"} & \text{"improper"} &
\end{tikzcd}$$

$$\mathcal{L} = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi, \quad S = \int d^4x\, \mathcal{L}$$

Parity

Can we find matrix A so that

$P\psi(t,\bar{x}) \to A\psi(t,-\bar{x})$ is a symmetry

of the Lagrangian? Note $\frac{d}{d\bar{x}}\psi(-\bar{x}) = -\frac{d}{d(-\bar{x})}\psi(-\bar{x})$

Because $\gamma^0\gamma^i = -\gamma^i\gamma^0$, this sign change is accomplished by $(\eta \in \mathbb{C}, |\eta| = 1)$

$$\begin{cases} P\psi(t,\bar{x}) \to \eta\gamma^0\psi(t,-\bar{x}) \quad \Rightarrow \\ P\bar{\psi}(t,\bar{x}) \to \eta^*\bar{\psi}(t,-\bar{x})\gamma^0 \end{cases} \qquad (4.55)$$

Charge conjugation

Consider "exchange" $\psi \Leftrightarrow \bar{\psi}$, or rather

$$\begin{cases} \psi_\alpha \to \bar{\psi}_\beta C^{-1}_{\beta\alpha} \\ \bar{\psi}_\alpha \to -C_{\alpha\beta}\psi_\beta \end{cases} \qquad (4.56)$$

Now $\mathcal{L} \to -C_{\alpha\beta}\psi_\beta(i\overrightarrow{\partial}_\mu[\gamma^\mu]_{\alpha\gamma} - m\,\delta_{\alpha\gamma})\bar{\psi}_\delta C^{-1}_{\delta\gamma}$

$$= +\bar{\psi}_\delta(i\overleftarrow{\partial}_\mu[C^{-1}_{\delta\gamma}(\gamma^{\mu T})_{\gamma\alpha}C_{\alpha\beta}] - m\,\delta_{\delta\beta})\psi_\beta$$

after partial integration, this becomes

$$S = \int d^4x[\bar{\psi}(i\overrightarrow{\partial}_\mu(-C^{-1}\gamma^{\mu T}C) - m)\psi]$$

Thus, we need $\underline{-C^{-1}\gamma^{\mu T}C = \gamma^\mu}$ (4.57)

Matrix $C = i\gamma^0\gamma^2$ satisfies this (homework!)

Time reversal

$T\psi(\bar{x},t) \to M\psi(\bar{x},-t)$ does not work for any M! We need to consider antiunitary transformations $T\psi(\bar{x},t) \to M\psi^*(\bar{x},-t)$, and we again can find M satisfying symmetry. $M = \gamma^1\gamma^3$ is one solution.

For free fermions, C, P, T are all symmetries. These are broken in the standard model by weak interactions (C, P maximally, combined C×P by a tiny bit). However, C×P×T should always be a symmetry of QFT:s!

5. Gauge field theories

5.1. U(1) Gauge field

- Maxwell equations are the oldest known physical field equations! They can be very compactly described by Lagrange density

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad (5.1)$$

where $\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$ is the (5.2) field tensor, A_μ gauge field or 4-potential, $A_\mu = (\underset{\substack{\uparrow \\ \text{scalar} \\ \text{pot., } V}}{A_0}, \underset{\substack{\nwarrow \\ \text{vector} \\ \text{potential}}}{\vec{A}})$

Eqn. of motion = Maxwell eqn.

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} - \frac{\delta \mathcal{L}}{\delta A_\nu} = 0 \Rightarrow \boxed{\partial_\mu F^{\mu\nu} = 0} \qquad (5.3)$$

Writing $\bar{B} = \nabla \times \bar{A}$ and $\bar{E} = -\nabla V - \partial_t \bar{A}$, we obtain

$$F^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & -E_x & -E_y & -E_z \\ \hline E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{array}\right)$$

Coupling to matter (bosons, fermions):
generalize ∂_μ to <u>covariant derivative</u>

$$\underbrace{\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu - i\underset{\uparrow}{e} A_\mu} \qquad (5.4)$$

charge, "coupling constant"

Applying this to Dirac eqn, we obtain the QED (Quantum electrodynamics) Lagrange density:

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \qquad (5.5)$$

Here $\not{D} \equiv \gamma^\mu D_\mu$. Extremizing wrt. A_μ and $\bar{\psi}$, we obtain eqns of motion

$$\begin{cases} \partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi \equiv j^\nu \\ (i\not{D} - m)\psi = 0 \end{cases} \qquad (5.6)$$

The 1st equation is Maxwell with current density j^μ ($j^0 = \rho$ charge density, $\vec{j}$ current)

Note: e dimensionless (when $\hbar = c = 1$), for electron charge $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$

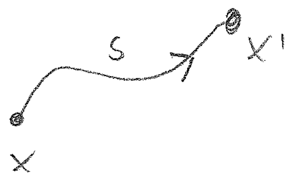
5.2. Gauge invariance

$\mathcal{L}_{QED}$ is invariant under a <u>local</u> transformation

$$\begin{cases} \psi \to e^{i\Lambda(x)}\psi \\ \bar{\psi} \to \bar{\psi}e^{-i\Lambda(x)} \\ A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \Lambda(x) \end{cases} \qquad (5.7)$$

where $\Lambda(x) \in \mathbb{C}$. This is a very large symmetry! This is called a gauge transformation.

A_μ describes how the phase angle of a complex field ϕ (or ψ) changes when we "parallel transport" it $x \to x'$

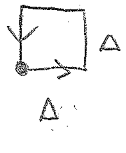


$$\varphi(x) \to \varphi(x') = e^{ie\int_x^{x'} ds^\mu A_\mu}\varphi(x)$$

$$= U_{x\to x'}\,\varphi(x)$$

Infinitesimally $\varphi \to (1 + ie\Delta s^\mu A_\mu)\varphi$

If we make parallel transport around an infinitesimal square on μ,ν-plane,



$$\varphi \to (1 + \Delta^2 F_{\mu\nu})\varphi$$

$F_{\mu\nu} = 0$: the complex phase does not change, "curvature" $= 0$

Compare: parallel transport of a tangent vector in curved space (e.g. surface of a sphere) around small square.

If $\underline{F_{\mu\nu} = 0}$ then A_μ is "pure gauge", i.e. $\exists\ \Lambda(x)$ so that $A_\mu = \frac{1}{e}\partial_\mu \Lambda(x)$, and we can gauge transform $A_\mu \to 0$.

How to obtain quantum field theory (QFT)?

Following earlier examples, we write path integral $\int DA_\mu D\psi D\bar{\psi}\, e^{iS}$.

Before going there, let us generalize the gauge transformations to larger non-commuting groups, specifically to SU(N)