5.3. Non-abelian gauge fields

- We can generalize the transformation

$$\phi \to e^{i\theta} \psi$$ of complex field to

a transformation of a complex vector:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N_c} \end{pmatrix} \in \mathbb{C}^{N_c} \qquad N_c: \text{"number of colors"}$$

$$\phi(x) \to \phi'(x) = U(x)\,\phi(x) \tag{5.8}$$

where $U(x)$ is a member of gauge group

- Demand that $|\phi|^2 = \phi^+\phi = \sum_i \phi_i^* \phi_i$ is invariant $\Rightarrow$

$$\phi^+\phi \to \phi^+ U^+ U \phi = \phi^+\phi \quad \text{for } \forall\,\phi$$

$$\Rightarrow \underline{U^+ U = \mathbb{1}} \qquad U \text{ is unitary matrix,}$$

$$\boxed{U \in U(N_c)} \quad \text{group of unitary } N_c^2 \text{ matrices.}$$

- Group:

1. If $A, B \in U(N_c)$, $C = AB \in U(N_c)$:

$$C^+ C = B^+ A^+ A B = \mathbb{1}.$$

2. For each $A \in U(N_c)$ $\exists\, A^{-1} \in U(N_c)$ so that $A^{-1}A = \mathbb{1}$:

$$A^{-1} = A^+ \in U(N_c)$$

3. $\mathbb{1} \in U(N_c)$

4. $(AB)C = A(BC)$

- Note: $A^\dagger A = \mathbb{1} \Rightarrow (\det A)^* \det A = 1$
$$\Rightarrow |\det A| = 1 \quad ; \quad A \in U(N_c)$$

Thus, we can always find a phase so that

$$A = e^{i\theta} B \quad , \quad \det B = 1 \quad , \quad B^\dagger B = B B^\dagger = \mathbb{1}$$

Here $\boxed{B \in SU(N_c)}$, group of special unitary matrices

$$U(N_c) = U(1) \otimes SU(N_c) \quad , \quad \text{direct product}$$

- The gauge groups appearing (or found, so far) in Nature are $U(1)$, $SU(2)$, $SU(3)$
QED WEAK QCD

- For fermions:

$$\psi_\alpha(x) \to \psi'_\alpha(x) = U(x)\,\psi_\alpha(x) \quad , \quad \alpha = 1..4 \text{ Dirac-index}$$

i.e. each component transforms separately;

$\psi_\alpha = \mathbb{G}^{N_c}$, N_c-comp. Grassmann-vector.

Full ψ is $4 \times N_c$-component Grassmann-vector

- Note:
$$\partial_\mu \phi(x) \to \partial_\mu (U(x)\phi(x))$$
$$= (\partial_\mu U)\,\phi + U\,\partial_\mu \phi$$
$$= U(U^\dagger \partial_\mu U + \partial_\mu)\,\phi$$

$\Rightarrow \partial_\mu \phi^\dagger \partial^\mu \phi$ is <u>not</u> gauge invariant!

Generalize $\partial_\mu \to D_\mu$, covariant derivative as in $U(1)$. But how?

- Any $U \in SU(N_c)$ can be written as

$$U = e^{i \sum_{a=1}^{N_c^2-1} \theta_a T_a} \tag{5.9}$$

$\theta_a \in \mathbb{R}$, $T_a = N_c \times N_c$-matrix, generators of $SU(N_c)$. $N_c^2 - 1$ different.

- $T_a^\dagger = T_a$ hermitean $(\Rightarrow U^\dagger = U^{-1})$ (5.10)
- $\mathrm{Tr}\, T_a = 0$ $(\Rightarrow \det U = 1)$ (5.11)
- $\mathrm{Tr}\,[T_a T_b] = \frac{1}{2} \delta_{ab}$ normalization (5.12)
- $[T_a, T_b] = \sum_c i f_{abc} T_c$ (5.13)

structure constants of the group Lie algebra

- For $SU(2)$, 3 generators:

$$T_a = \frac{1}{2} \sigma_a \quad ; \quad [T_a, T_b] = i \epsilon_{abc} \sigma_c \tag{5.14}$$

Pauli matrix

- For $SU(3)$, 8 Gell-Mann matrices

$$\lambda_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

• Why N_c^2-1 ? • Clearly, we can write $U = e^{iA}$

$$A^\dagger = A, \quad \mathrm{tr}\, A = 0 \qquad \left(\det M = e^{\mathrm{tr} \ln M}\right)$$

• $A^\dagger = A \Rightarrow \begin{cases} a_{ii} \in \mathbb{R} & \text{(no sum)}, \; N_c \text{ real} \\ a_{ij} = a_{ji}^* & \frac{1}{2}N_c(N_c-1)\times 2 \text{ real} \end{cases}$

real degrees of freedom

• $\mathrm{tr}\, A = 0 \Rightarrow \sum_i a_{ii} = 0$ takes away 1 real d.o.f.

$$\Rightarrow N_c(N_c-1) + N_c - 1 = \underline{N_c^2-1}$$

Try now covariant derivative

$$\boxed{D_\mu \equiv \partial_\mu - igA_\mu} \qquad (5.15)$$

$g \equiv$ coupling constant

$$\underline{A_\mu(x) = \sum_{a=1}^{N_c^2-1} A_\mu^a(x)\, T^a}, \quad A_\mu^a(x) \in \mathbb{R} \qquad (5.16)$$

Now, transformation

$$\boxed{A_\mu(x) \longrightarrow A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)} \qquad (5.17)$$

Leads to gauge invariance under (5.8)!

Proof :

a) Transformation keeps properties of A:

- $$\mathrm{Tr}\,A' = \mathrm{Tr}\,UAU^+ + \frac{i}{g}\mathrm{Tr}\,U\partial_\mu U^+$$

$$= \mathrm{Tr}\,\underbrace{U^+U}_{1}A + \frac{i}{g}\mathrm{Tr}\,\underbrace{\partial_\mu U^+}_{-i(\partial_\mu\theta^a)T^a U^+}\cdot U \quad (\text{from } (5.9))$$

$$= \mathrm{Tr}\,A = 0. \tag{5.18}$$

- $$A'^+ = UA^+U^+ - \frac{i}{g}\underbrace{(\partial_\mu U)U^+}_{\partial_\mu(UU^+) - U\partial_\mu U^+} = A' \quad \text{ok.}$$

$$\partial_\mu(UU^+) = 0$$

b) $$D_\mu'\phi' = (\partial_\mu - igUA_\mu U^+ + U\partial_\mu U^+)\cdot U\phi$$

$$= U(\partial_\mu - igA_\mu)\phi + \underbrace{((\partial_\mu U)U^+ - U\partial_\mu U^+)}_{=0}U\phi$$

$$= U D_\mu\phi \tag{5.19}$$

$$\Leftrightarrow \boxed{D_\mu' = U D_\mu U^+} \tag{5.20}$$

Thus, constructs like

$$\mathcal{L}_M = (D^\mu\phi)^+(D_\mu\phi) - V(\phi^+\phi) \tag{5.21}$$

$$\mathcal{L}_M = \bar\psi_\alpha\left\{i[\gamma^\mu]_{\alpha\beta}D_\mu - m\,\delta_{\alpha\beta}\right\}\psi_\beta \tag{5.22}$$

$$\equiv \bar\psi(i\gamma^\mu D_\mu - m)\psi$$

are gauge invariant and suitable for $\mathcal{L}$ (for scalars and fermions)

How about the kinetic term for A_μ?

- Define

$$\boxed{F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] = F^a_{\mu\nu} T^a} \tag{5.23}$$

which is, in component form,

$$\begin{aligned} F^a_{\mu\nu} &= 2\,\mathrm{Tr}[T^a F_{\mu\nu}] \\ &= \frac{2i}{g}\mathrm{Tr}\Big[T^a\Big(\underbrace{[\partial_\mu, \partial_\nu]}_{=0} - ig\underbrace{[\partial_\mu, A_\nu]}_{=\partial_\mu A^b_\nu T^b} - ig\underbrace{[A_\mu, \partial_\nu]}_{=-\partial_\nu A^b_\mu T^b} - g^2\underbrace{[A_\mu, A_\nu]}_{=A^b_\mu A^c_\nu i f^{bcd} T^d}\Big)\Big] \\ &= \boxed{\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu} \end{aligned} \tag{5.24}$$

- Or, alternatively,

$$\underline{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]} \tag{5.25}$$

- Here we used $f^{abc} = f^{cab}$; f^{abc} antisymmetric w.r.t. any index swap

- In gauge transformations

$$\underline{F'_{\mu\nu}} = \frac{i}{g}[D'_\mu, D'_\nu] \underset{(5.20)}{=} \frac{i}{g}[U D_\mu U^\dagger, U D_\nu U^\dagger] = \underline{U F_{\mu\nu} U^\dagger} \tag{5.26}$$

Thus, $\underline{\frac{1}{2}\mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}]} = \underline{\frac{1}{4} F^{a\,\mu\nu} F^a{}_{\mu\nu}}$ is gauge-invariant. (5.27)

Thus, we obtain (with "canonical" normalization)

$$\mathcal{L} = -\frac{1}{4} F^{a\,\mu\nu} F^{a}{}_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

with $N_c = 3$: <u>QCD</u>
$(a = 1,2,3)$

$$\mathcal{L} = -\frac{1}{4} F^{a\,\mu\nu} F^{a}{}_{\mu\nu} + (D^\mu \phi)^\dagger (D_\mu \phi) - V(\phi^\dagger \phi)$$

$N_c = 2$:

<u>weak gauge field</u>
<u>+ Higgs boson</u>

And with $N_c \to 1$, $T^a \to 1$, $A_\mu^a \to A_\mu$

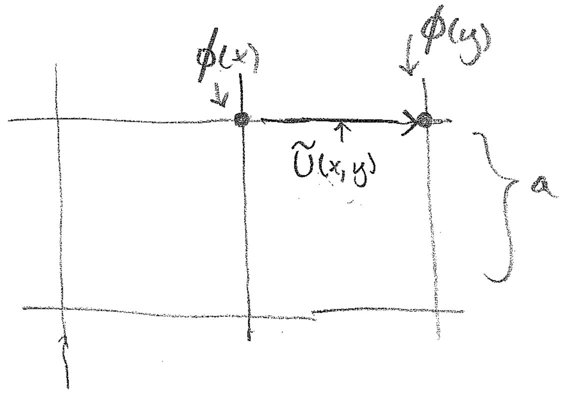
$F_{\mu\nu}^a \to F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ we obtain <u>QED</u>:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

NOTE: the defining feature of the covariant derivative is the gauge transformation (5.20) : $D_\mu' = U D_\mu U^+$. With this we can construct gauge invariant terms $\phi^+ D_\mu \phi$, $(D_\mu \phi)^+ (D^\mu \phi)$. (5.20) and (5.15) $\Rightarrow$ (5.17), $A_\mu' = U A_\mu U^+ + \frac{i}{g} U(\partial_\mu U^+)$

Gauge fields on discrete lattice

An intuitive (and, again, mathematically solid) framework for gauge invariance is provided by discrete lattice:



approximate now

$$\partial_\mu \phi \cong \frac{1}{a}(\phi(y) - \phi(x)). \quad (5.23)$$

However, if we have local gauge transformation,

$\phi(x) \to U(x)\phi(x)$; $\phi(y) \to U(y)\phi(y)$, $U(x), U(y)$ arbitrary,

the finite difference does not make sense.

However, if we postulate a "gauge connection"

("link matrix") $\tilde{U}(x,y) \in$ (Gauge group), which "parallel transports" $\phi(y)$ to x and which transforms as

$$\tilde{U}(x,y) \Rightarrow U(x)\tilde{U}(x,y)U^+(y) \quad (5.24)$$

we can define a covariant finite difference

$$D_\mu \phi(x) \cong \frac{1}{a}(\tilde{U}(x,y)\phi(y) - \phi(x)) \quad (5.25)$$

Now $D_\mu \phi(x) \to U(x) D_\mu \phi(x)$, and $(D_\mu \phi)^+ (D^\mu \phi)$ is gauge invariant.

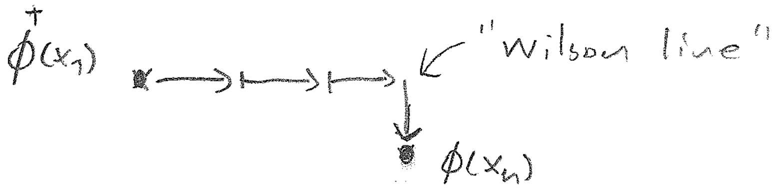
This corresponds to parallel transport on p. 120

Infinitesimally $\tilde{U} = e^{-iaA}$, and (5.24) $\Rightarrow$

$$A_\mu \to U A_\mu U^+ + \frac{i}{g} U \partial_\mu U^+$$

It is now easy to see that the only gauge invariant quantities are

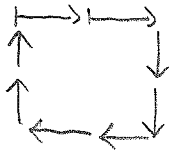
$$\phi^\dagger(x_1)\,\tilde{U}(x_1,x_2)\,\tilde{U}(x_2,x_3)\cdots\tilde{U}(x_{n-1},x_n)\,\phi(x_n)$$



and closed lines

$$\mathrm{Tr}\left[\tilde{U}(x_1,x_2)\cdots\tilde{U}(x_{n-1},x_1)\right]$$

(Wilson loop)



Smallest loop, "plaquette"

 $= 1 + i a^2 F^a_{\mu\nu} T^a + \mathcal{O}(a^4)$

$\longrightarrow$ Lattice QCD, lattice field theory

5.4. Quantization and gauge fixing

We can now define the path integral

$$Z = \int DA_\mu \, D\psi \, D\bar{\psi} \; e^{iS} \qquad (5.26)$$

which defines the quantum theory. This is fully gauge invariant. (This is done in numerical simulations)

However, the gauge invariance causes problems for analytical treatment of path integrals. For example, we cannot even define the Feynman propagator for A_μ.

Or, consider U(1) Lagrange density:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

A_μ: canonical fields ?

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} = F_{0\mu} = E_\mu, \quad \underset{\sim}{\mu \neq 0} \qquad (5.27)$$

what is canonical momentum for A_0?

Does not exist. Or rather, set A_μ, $\mu = 0..3$ contains too many d.o.f.'s. Because of the gauge symmetry, we can set one of them to any value we choose.

$\Rightarrow$ Gauge fixing: fix the gauge using some condition.