Faddeev-Popov method (U(1) gauge field)

- Let $G(A)=0$ be the condition used to fix the gauge:

$$A_0 = 0 \quad \text{temporal gauge}$$
$$\partial_\mu A^\mu = 0 \quad \text{Lorentz gauge} \qquad (5.28)$$
$$\partial_i A^i = 0 \quad \text{Coulomb gauge etc.}$$

- Fix it: $\int DA_\mu(x) \longrightarrow$ (5.29)

$$\int DA_\mu(x)\ \delta(G(A)) \times \det\left[\frac{\delta G}{\delta \theta}\right]$$

Gauge transformation

Fixes the gauge
$\delta(G(A)) \equiv \prod_x \delta(G(A(x)))$

Faddeev-Popov determinant, corrects the measure (~ Jacobian)

Here θ parametrizes gauge transformations, i.e.

$$U = e^{ig\theta}, \qquad A_\mu \to A'_\mu = A_\mu + \partial_\mu \theta$$

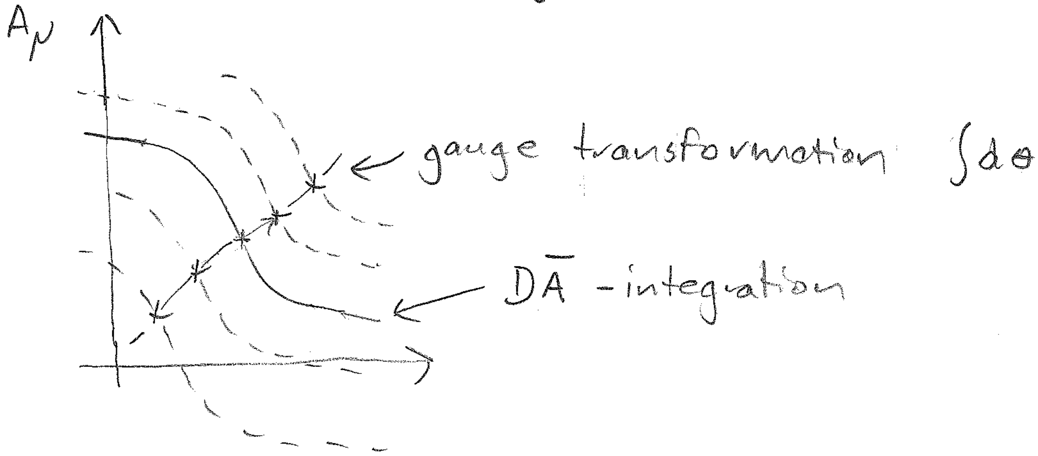
(c.f. 5.7)

- Why Faddeev-Popov det?

Our aim is to integrate over gauge fields $A_\mu(x)$ which are **not** gauge-equivalent, or cannot be gauge transformed into each other.

Divide $\int DA_\mu$ into 2 parts;

- $\int D\bar{A}_\mu$ - gauge non-equivalent configs.
- $\int d\theta$ - gauge transformations



Now
$$\int DA_\mu \, \delta(G) \det\left[\frac{\delta G}{\delta \theta}\right] =$$
$$\int D\bar{A}_\mu \int d\theta \; \delta(G) \det\left[\frac{\delta G}{\delta \theta}\right] =$$
$$\int D\bar{A}_\mu \int dG \; \delta(G) = \int D\bar{A}_\mu \; ! \qquad (5.30)$$

This would not work without det.

Important result:

$$\int DA_\mu \, \delta(G) \det\left[\frac{\delta G}{\delta \theta}\right] F(A_\mu) \qquad (5.31)$$

is independent of the choice of G, if $F(A_\mu)$ is gauge invariant!

Thus,

$$\langle \ldots \rangle \equiv \int DA_\mu \, \delta(G) \det\left[\frac{\delta G}{\delta \theta}\right] (\ldots) \, e^{iS} \qquad (5.32)$$

gauge invariant observable

is also gauge invariant.

Faddeev-Popov ghosts

- Instead of gauge fixing condition $G(A)=0$ we can as well use $G(A(x)) - \omega(x) = 0$, where $\omega(x)$ is arbitrary scalar function. This is still a valid gauge! We can now integrate over $\omega(x)$ with a gaussian weight

$$\delta(G) \longrightarrow \int D\omega \; \delta(G-\omega)\, e^{-i\int d^4x \frac{1}{2\xi}\omega(x)^2} \tag{5.33}$$

$$= e^{-i\int d^4x \frac{1}{2\xi} G^2(A)} \tag{5.34}$$

$\xi \geq 0$ is a free gauge parameter; physical quantities must be independent of it!

($\xi \to \infty$: no gauge fixing; $\xi \to 0$: $\delta(G)$ gauge)

- Recall : $\det M = \int Dc\, D\bar{c}\, e^{-\bar{c}Mc}$ Grassmann

Thus, we can formally write

$$\det\left[i\frac{\delta G}{\delta\theta}\right] = \int Dc\, D\bar{c}\, e^{i\int d^4x\; \bar{c}\left(\frac{\delta G}{\delta\theta}\right)c} \tag{5.35}$$

$\det iA = i^N \det A$; i^N does not matter in P.I.

This is an integral over Grassmann variables; however, $c, \bar{c}$ is "scalar" (not spinor). spin-0 fermions ?

These ghosts do not correspond to physical d.o.f.'s.

We obtain

$$\langle \ldots \rangle = \int \mathcal{D}A_\mu \int \mathcal{D}c\, \mathcal{D}\bar{c}\, (\ldots)\, e^{i\int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi} G^2 + \bar{c} \frac{\delta G}{\delta \theta} c \right\}} \tag{5.36}$$

Examples :

a) $G(A) = A_0 = 0$

$$\frac{\delta G}{\delta \theta} = \frac{\delta(A_0 + \partial_0 \theta)}{\delta \theta} = \partial_0$$

$$\mathcal{L}_{Ghost} = \bar{c}\, \partial_0 c$$

b) $G(A) = \partial_\mu A^\mu = 0$

$$\frac{\delta G}{\delta \theta} = \frac{\delta}{\delta \theta} \left(\partial_\mu (A^\mu + \partial^\mu \theta) \right) = \partial_\mu \partial^\mu \tag{5.37}$$

$$\Rightarrow \mathcal{L}_{Ghost} = \bar{c}\, \partial_\mu \partial^\mu c = -\partial_\mu \bar{c}\, \partial^\mu c$$

partial integration

Note: for U(1) (QED), ghosts decouple from A_μ's and can be ignored. Not so for non-abelian gauge theory!

For <u>non-abelian</u> theory (QCD) the gauge condition is for all colours:

$$G^a \equiv \partial_\mu A^{a\mu} = 0 \qquad (5.38)$$

$\uparrow$ covariant gauge

The generalization of gauge fixed path integral is

$$Z = \int \mathcal{D}A_\mu \int \mathcal{D}\bar{c}\,\mathcal{D}c \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi} \times$$

$$\exp\left[i\int d^4x \left\{ -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2\xi} G^a G^a + \bar{c}^a \left(\frac{\delta G^a}{\delta \theta^b} \right) c^b \right\} \right] \qquad (5.39)$$

- $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$
- $\not{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - ig A^a_\mu T^a)$
- $\frac{1}{2\xi} G^a G^a = \frac{1}{2\xi} \partial_\mu A^{a\mu} \partial_\nu A^{a\nu}$ in covariant gauge
- Now $A'_\mu = U A_\mu U^+ + \frac{i}{g} U \partial_\mu U^+$

Parametrizing $U = e^{ig\theta^a T^a} \equiv e^{ig\theta}$

$$U A_\mu U^+ = A_\mu + ig[\theta, A_\mu] + O(\theta^2)$$
$$= T^a \{ A^a_\mu - g f^{abc} \theta^b A^c_\mu \} + O(\theta^2)$$

$$\frac{i}{g} U \partial_\mu U^+ = -\frac{i}{g} \partial_\mu U U^+ = \partial_\mu \theta^a T^a + O(\theta^2)$$

Thus, now

$$A'^a_\mu = A^a_\mu + \partial_\mu \theta^a + g f^{abc} A^b_\mu \theta^c + \mathcal{O}(\theta^2) \qquad (5.40)$$

$$\Rightarrow \quad \frac{\delta G^a}{\delta \theta^b} = \partial_\mu \frac{\delta A^{a\mu}}{\delta \theta^b}$$

$$= \partial_\mu \left(\partial^\mu \delta^{ab} + g f^{acb} A^{c\mu} \right) \qquad (5.41)$$

$$\Rightarrow \quad \bar{c}^a \left(\frac{\delta G^a}{\delta \theta^b} \right) c^b = -\partial_\mu \bar{c}^a \partial^\mu c^a - \partial_\mu \bar{c}^a g f^{abc} A^{b\mu} c^c \qquad (5.42)$$

$$(\bar{c} \partial_\mu \partial^\mu c \rightarrow -\partial_\mu \bar{c} \partial^\mu c)$$

Thus, in this case there is interaction between ghosts $\bar{c}, c$ and A_μ !

Ghosts can appear as internal lines in diagrams
Note: ghost action (& interactions) depend on gauge choice

Propagators and interactions

Easiest to find if we consider the part quadratic in fields (A, ψ, c) and FT it: (5.39):

$$S^{(2)} = \int d^4x \Big[-\tfrac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})$$

$$- \frac{1}{2\xi} \partial_\mu A^{a\mu} \partial_\nu A^{a\nu} + \bar\psi^a (i\gamma\!\!\!/ - m)\psi^a - \partial_\mu \bar c^a \partial^\mu c^a \Big] \quad (5.43)$$

$$= \int d^4x \Big[\tfrac{1}{2} A^a_\mu \left(\partial_\alpha \partial^\alpha g^{\mu\nu} - \partial^\mu \partial^\nu \left(1 - \tfrac{1}{\xi}\right)\right) A^a_\nu$$

$$+ \bar\psi^a (i\partial\!\!\!/ - m)\psi^a - \partial_\mu \bar c^a \partial^\mu c^a \Big]$$

$A \in \mathbb{R}, -p$

$$= \int \frac{d^4p}{(2\pi)^4} \Big[\tfrac{1}{2} \tilde A^a_\mu(p) \left(-p^2 g^{\mu\nu} + p^\mu p^\nu \left(1 - \tfrac{1}{\xi}\right)\right) \tilde A^a_\nu(-p)$$

$$+ \tilde{\bar\psi}^a(p)(p\!\!\!/ - m)\tilde\psi^a(p) + p^2 \tilde{\bar c}^a \tilde c^a \Big] \quad (5.44)$$

The propagators can be recognized from above

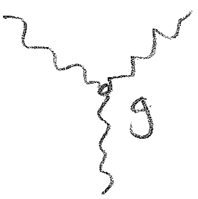
$\tfrac{1}{2} A(p) (i\tilde D_F)^{-1} A(-p)$; $\tilde{\bar\psi}(p)(i\tilde S_F)^{-1} \psi(p)$; $\tilde{\bar c}(p)(i\tilde G_F)^{-1} \tilde c(p)$

$$\boxed{\begin{aligned} \tilde D_F^{\mu\nu, ab} &= \frac{-i}{p^2 + i\epsilon}\left[g^{\mu\nu} - (1-\xi) p^\mu p^\nu \right] \delta^{ab} && (5.45) \\ \tilde S_F^{ab} &= \frac{i}{p\!\!\!/ - m + i\epsilon} \delta^{ab} = i \frac{p\!\!\!/ + m}{p^2 - m^2 + i\epsilon} \delta^{ab} && (5.46) \\ \tilde G_F^{ab} &= \frac{i}{p^2 + i\epsilon} \delta^{ab} && (5.47) \end{aligned}}$$

Interactions arise from 3- and 4-field terms (without details here)

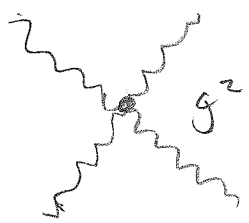
- Commutator $[A_\mu, A_\nu]$ in $F_{\mu\nu}$ gives rise to terms

$\sim g\, \partial A \cdot [A, A]$



3-gluon vertex

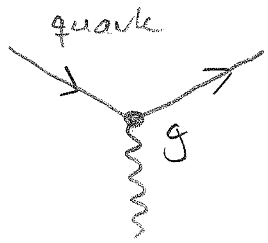
$\sim g^2 [A, A]^2$



4-gluon vertex

- $\bar\psi \not{D} \psi \rightarrow$

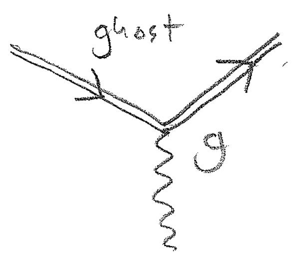
$\sim g\, \bar\psi \gamma^\mu \psi A_\mu$



quark-gluon vertex

- Ghost

$\sim g\, \partial \bar{c}\, c\, A$



ghost-gluon vertex

When we go to QED, only quark-gluon vertex survives!

- $\Rightarrow$ Feynman rules