Note: if $\phi(x) \in \mathbb{C}$,

$$\begin{cases} \tilde{\phi}(k) = \int d^4x \, \phi(x) \, e^{ik\cdot x} \\ \phi(x) = \int \frac{d^4k}{(2\pi)^4} \, \tilde{\phi}(k) \, e^{-ik\cdot x} \end{cases} \qquad (5.48)$$

and

$$\int d^4x \, \phi^*(x) \, M(\partial_x) \, \phi(x) = \int \frac{d^4k}{(2\pi)^4} \, \tilde{\phi}^*(k) \, M(-ik) \, \tilde{\phi}(k) \qquad (5.49)$$

But if $\varphi(x) \in \mathbb{R}$, $\underline{\tilde{\varphi}(k)^* = \tilde{\varphi}(-k)}$ and (5.50)

$$\frac{1}{2} \int d^4x \, \varphi(x) \, M(\partial_x) \, \varphi(x) = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \, \tilde{\varphi}(k) \, M(ik) \, \tilde{\varphi}(-k) \qquad (5.51)$$

Factor of $\frac{1}{2}$ in front of real-valued lagrangian is conventional.

- Scalar field (p. 35)

$$\int d^4x \left(\frac{1}{2} \partial_\nu \varphi \partial^\nu \varphi - \frac{1}{2} m^2 \varphi^2 \right) = \int d^4x \, \varphi \left(-\frac{1}{2} \partial_\nu \partial^\nu - \frac{1}{2} m^2 \right) \varphi$$

$$= \int \frac{d^4k}{(2\pi)^4} \, \tilde{\varphi}(k) \underbrace{\frac{1}{2}(k^2 - m^2)}_{(\tilde{G}_F(k))^{-1} \quad (1.45)} \tilde{\varphi}(-k)$$

5.6. Feynman rules for QED

• In QED the propagators are (from p. 138)

$$\tilde{D}_F^{\mu\nu} = \frac{-i}{p^2+i\epsilon}\left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

$$\tilde{S}_F = \frac{i}{\not{p}-m+i\epsilon} = i\frac{\not{p}+m}{p^2-m^2+i\epsilon} \qquad (5.52)$$

and the ghost propagator is not needed!

• $$\begin{cases} \xi = 1 : \ \underline{\text{Feynman gauge}} \\ \xi = 0 : \ \underline{\text{Landau gauge}} \end{cases} \qquad (5.53)$$

are popular gauge choices. ($\xi = \infty$: no gauge fixing!)

• In x-space,

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{D}^{\mu\nu}(k)\, e^{-ik\cdot(x-y)} \qquad (5.54)$$

obeys

$$\left[\partial_\rho \partial^\rho g_{\mu\nu} - \partial_\mu \partial_\nu \left(1-\frac{1}{\xi}\right)\right] D_F^{\nu\sigma}(x-y) = i g_\mu{}^\sigma\, \delta^{(4)}(x-y)$$

as it should.

Note that

$$\tilde{D}_F^{\mu\nu} = \frac{-i}{p^2+i\epsilon}\left(P_T^{\mu\nu} + \xi\, P_L^{\mu\nu}\right) \tag{5.55}$$

where $P_T^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}$; $P_L^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}$ (5.56)

are <u>transverse</u> and <u>longitudinal</u> projectors, corresponding to modes $\perp \bar{p}$ and $\parallel \bar{p}$.

These obey

$$P_T^2 = P_T ; \quad P_L^2 = P_L ; \quad P_T P_L = P_L P_T = 0$$
$$P_L + P_T = 1 \tag{5.57}$$

Plane waves of photons

We have not yet looked at free photon solutions. The wave equation is

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0 \tag{5.58}$$

Let us apply <u>covariant gauge</u>

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \tag{5.59}$$

$$\partial_\mu \partial^\mu A^\nu = 0$$

Plane wave ansatz (generalizing to complex A)

$$A^\mu = \epsilon^\mu e^{-ik\cdot x} \tag{5.60}$$

with complex ϵ^μ. $\Rightarrow$

$$-k^2 A^\mu = 0 \quad \Rightarrow \quad k^2 = k^{0\,2} - \vec{k}^2 = 0 \tag{5.61}$$

"lightlike"

$$-ik\cdot\epsilon\, e^{-ik\cdot x} = 0 \quad \Rightarrow \quad k\cdot\epsilon = 0 \tag{5.62}$$

or ϵ^μ is transverse. Formally ϵ has 4 d.o.f's, but only 2 are physical.

choice:
$$\begin{cases} \epsilon^{(1,2)} = (0, \vec{e}), & \vec{e} \perp \vec{k} \\ \epsilon^{(3)} = (0, \hat{k}) \\ \epsilon^{(0)} = (1, \vec{0}) \end{cases}$$

$\epsilon^{(0)}$ and $\epsilon^{(3)}$ are related through $k\cdot\epsilon = 0$. In addition, $\epsilon^{(0)}$ and $\epsilon^{(3)}$ are not independent (later). Polarizations obey completeness relation

$$\epsilon^{(0)*}_\mu \epsilon^{(0)}_\nu - \sum_{\lambda>0} \epsilon^{(\lambda)*}_\mu \epsilon^{(\lambda)}_\nu = g_{\mu\nu} \tag{5.63}$$

(needed for external legs in diagrams)

Polarizations are more commonly studied in Coulomb gauge

$$\partial_i A^i = 0 \quad \Rightarrow \quad \vec{k}\cdot\vec{\epsilon} = 0$$

In this case

$$\partial_i F^{i0} = \partial_i \partial^i A^0 - \cancel{\partial_i \partial^0 A^i} = 0$$

$$\Rightarrow \nabla^2 A^0 = 0$$

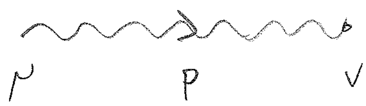
$\Rightarrow$ A^0 does not obey wave equ, but is determined by boundary conditions (sources at boundary). If no sources, $A^0 = 0$.
In this case, $\epsilon^{(1,2)} = (0, \vec{e})$, $\vec{e} \perp \vec{k}$ are physical polarizations - transverse.

(in Coulomb gauge,

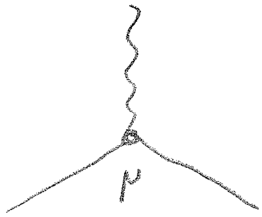
$$\sum_{\lambda=1,2} \epsilon^{(\lambda)*}_{\mu} \epsilon^{(\lambda)}_{\nu} = g_{\mu\nu} - \frac{1}{k^{0 2}}\left(k_\mu k_\nu + k^0 (k_\mu \delta_{\nu,0} + k_\nu \delta_{\mu,0})\right)\Big)$$

Pulling all of this together, we can obtain QED Feynman rules:

$\frac{i}{\not{p}-m+i\epsilon}$ fermion propagator

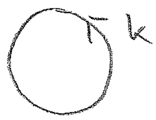


$\frac{-i g^{\mu\nu}}{p^2+i\epsilon}$ photon prop. in Feynman gauge ($\xi=1$)



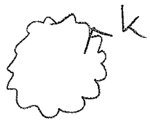
$-ie\gamma^\mu$ interaction vertex (5.4), (5.5)

($\times$ 4-mom. conservation!)



$\int \frac{d^4k}{(2\pi)^4} \cdot (-\mathrm{Trace}(\gamma's))$

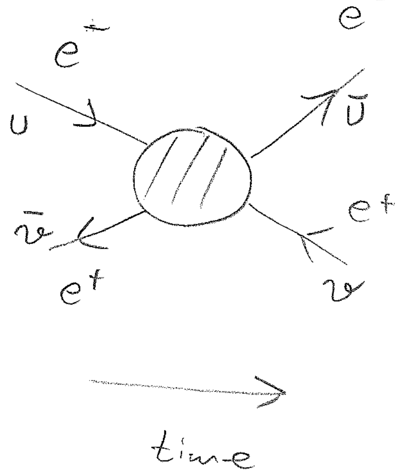
for each closed fermion loop



$\int \frac{d^4k}{(2\pi)^4}$ for photon loops

- External fermions: (e, for concreteness)

	incoming	outgoing
e^- (fermion)	$u(p,s)$	$\bar{u}(p,s)$
e^+ (antifermion)	$\bar{v}(p,s)$	$v(p,s)$

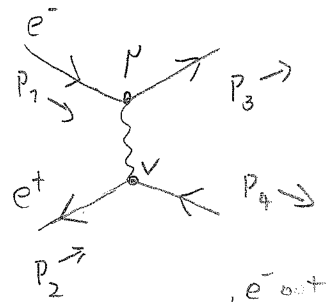


- External photons:

  incoming ϵ_μ, outgoing ϵ^*_μ

- Fermion sign (later!)
- Overall $(+i)$

Example: e^+e^- "t-channel"



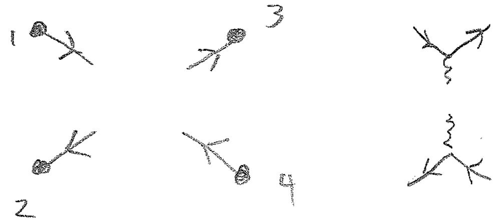
- Follow fermion lines agaist the arrows

e^- out $\qquad$ e^- in

$$= i\,\left(\bar u(p_3)\,(ie\gamma^\mu\,u(p_1)\right)\frac{-ig_{\mu\nu}}{(p_1-p_2)^2+i\epsilon}\times$$

$$\left(\bar v(p_2)\,(ie\gamma^\nu)\,v(p_4)\right)$$

Symmetry factor — connect vertices & legs:



$$\frac{2\cdot1\cdot1\cdot1}{2!} = 1$$

$\uparrow$ 2 vertices

leg 1 can be connected to 2 vertex lines