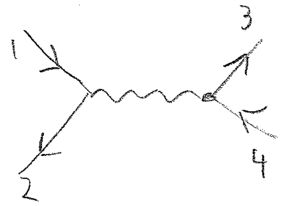
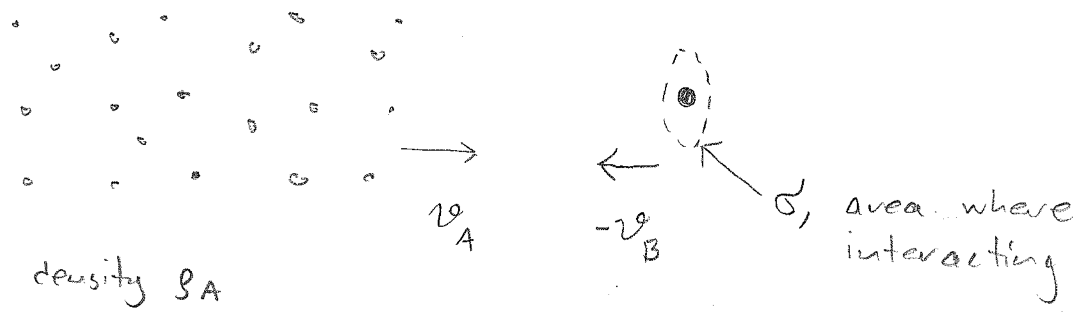
in s-channel



$$i\,(\bar{v}(p_2)(ie\gamma^\mu)u(p_1))\,\frac{-ig_{\mu\nu}}{(p_1+p_2)^2+i\epsilon}\,(\bar{u}(p_3)(ie\gamma^\nu)v(p_4)) \quad (5.67)$$

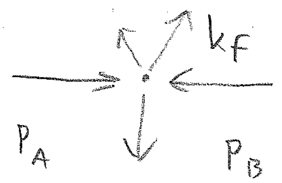
5.7. Cross-section and S-matrix (part. phys. notes)

- Cross-section: effective "area" where interactions occur: $A+B\to$



- Differential cross-section

  Scattering to a specific state



$$\frac{d\sigma}{\prod_f dk_f}\;;\quad \sigma=\int dk_F\,\frac{d\sigma}{dk_F}$$

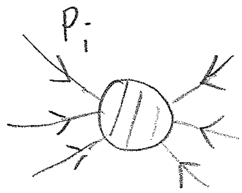
- S-matrix

From particle physics notes (p. 115) we recall the definition of S-matrix:

$$S_{FI} = -i \langle \text{Final} \,|\, \int dt\, \hat{V}_I \,|\, \text{Initial} \rangle \tag{5.68}$$

interaction

$$= -i (2\pi)^4 \delta^{(4)}\Big(\sum_i p_i\Big) \frac{\mathcal{M}}{\prod_i \left[(2\pi)^3 2E_{\vec{p}_i}\right]^{1/2}} \tag{5.69}$$

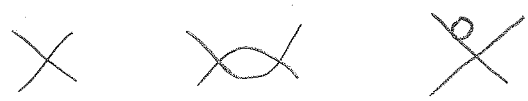


$P_{In} = P_{out}$

i labels all particles, in or out.

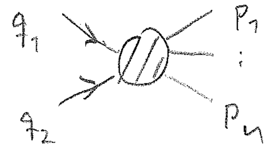
- Amplitude $\mathcal{M}$ $= -i \sum \begin{pmatrix} \text{Amputated, 1PI} \\ \text{Feynman diagrams} \end{pmatrix}$ (5.70)

Example:



OK   OK   NOT 1PI!

- Golden rule for scattering: (p. 123)
  - Consider $2 \to n$



$$\sigma = \frac{S}{F} \int d\Phi_n |\mathcal{M}|^2 \tag{5.71}$$

where

$$F = 4\sqrt{(q_1 \cdot q_2)^2 - (m_1 m_2)^2} \qquad (5.72)$$

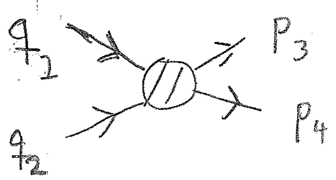
(flux factor), and

$$S = \prod_{\substack{j = \text{kinds of} \\ \text{particles}}} 1/n_j! \quad ; \quad \sum_j n_j = n \qquad (5.73)$$

and the phase space integral

$$d\Phi_n = \left[\prod_{i=1}^{n} \frac{d^3\bar{p}_i}{(2\pi)^3 2E_i}\right] (2\pi)^4 \, \delta^{(4)}\left(q_1 + q_2 - \sum_i{}' p_i\right) \qquad (5.74)$$

• In $2 \to 2$ scattering in center-of-mass frame this simplifies considerably: $\bar{q}_2 = -\bar{q}_1$, $\bar{p}_4 = -\bar{p}_3$



$$E_{TOT} = E_1 + E_2 = E_3 + E_4$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{|\bar{p}_3|}{|\bar{q}_1|} \frac{1}{E_{TOT}^2} |\mathcal{M}|^2 \qquad (5.75)$$

Here Ω = space angle of $\bar{p}_3$. In this case all other $\bar{p}_3, \bar{p}_4$-integrals can be done independently of $\mathcal{M}$.

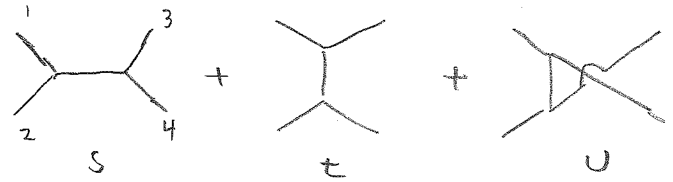
- $\mathcal{M}$: interactions, "physics"
- rest: "kinematics", geometry

Example: $\mathcal{L}_I = -\frac{1}{4!}\lambda\varphi^4$, lowest order

$\mathcal{M} =$  $= -i\lambda \quad (+\mathcal{O}(\lambda^2))$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^2}{64\pi^2 E_{cm}} \quad ; \quad E_{CM} = 2\sqrt{p^2+m^2}$$

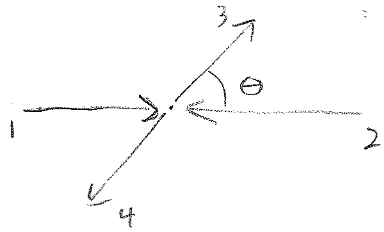
Another: $\mathcal{L}_I = -\frac{1}{3!}g\varphi^3$, $2\to2$ to order g^2:

$\mathcal{M} =$ 

"Mandelstam variables"

$$= (-i\lambda)^2\left(\frac{i}{(p_1+p_2)^2-m^2} + \frac{i}{(p_1-p_3)^2-m^2} + \frac{i}{(p_1-p_4)^2-m^2}\right)$$

CM-frame: $\bar{p}_2 = -\bar{p}_1 \Rightarrow (p_1+p_2)^2 = 4p_1^{0\,2} = 4E_1^2$



$$(p_1-p_3)^2 = -(\bar{p}_1-\bar{p}_3)^2 = -2p^2 + 2\bar{p}_1\cdot\bar{p}_3$$
$$= 2p^2(1-\cos\theta)$$

$$(p_1-p_4)^2 = -2p^2(1+\cos\theta)$$

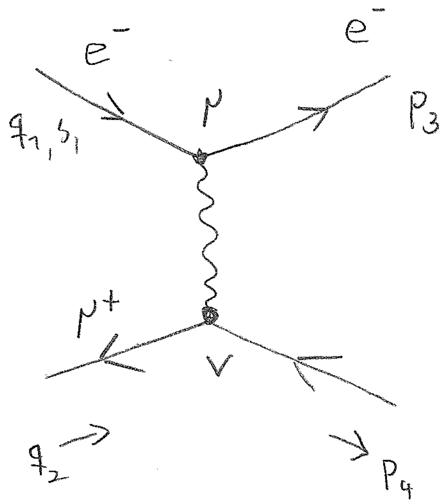
$$\mathcal{M} = -i\lambda^2\left(\frac{1}{4p^2+3m^2} + \frac{-1}{2p^2(1-\cos\theta)+m^2} + \frac{-1}{2p^2(1+\cos\theta)+m^2}\right)$$

Let us set $m=0$ (ultrarel. limit), for simplicity:

$$\mathcal{M} \to i\lambda^2\,\frac{\sin^2\theta-4}{4p^2\sin^2\theta}$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{1}{(8\pi)^2}\frac{1}{4p^2}|\mathcal{M}|^2$$

Example : $e^- \mu^+ \to e^- \mu^+$



Follow each fermion line backwards:

$$\mathcal{M} = i\, \underbrace{\bar{u}(p_3,s_3)}_{e^- \text{ out}}\, ie\gamma^\mu\, \underbrace{u(q_1,s_1)}_{e^- \text{ in}} \times$$

$$\frac{-ig_{\mu\nu}}{(q_1-p_3)^2} \times \underbrace{\bar{v}(q_2,s_2)}_{\mu^+ \text{ in}}\, ie\gamma^\nu\, \underbrace{v(p_4,s_4)}_{\mu^+ \text{ out}}$$

$$= \frac{-e^2}{(q_1-p_3)^2}\, \bar{u}(p_3,s_3)\gamma^\mu u(q_1,s_1) \times \bar{v}(q_2,s_2)\gamma_\mu v(p_4,s_4) \qquad (5.76)$$

Things simplify when we are not interested about the spin: average over initial, sum over final spins:

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2}\frac{1}{2} \sum_{s_1 s_2 s_3 s_4} |\mathcal{M}|^2 \qquad (5.77)$$

$$|\mathcal{M}|^2 = \frac{e^2}{(q_1-p_3)^2} [\bar{u}(3)\gamma^\mu u(1)][\bar{u}(3)\gamma^\nu u(1)]^*$$
$$[\bar{v}(2)\gamma_\mu v(4)][\bar{v}(2)\gamma_\nu v(4)]^*$$

$$(5.78)$$

Now $[\bar{u}(3)\gamma^\nu u(1)]^* = [u(3)^\dagger\gamma^0\gamma^\nu u(1)]^\dagger$

$= [u(1)^\dagger \gamma^{\nu\dagger}\gamma^{0\dagger} u(3)]$ $\qquad \gamma^{\nu\dagger} = \gamma^0\gamma^\nu\gamma^0$

$= [u(1)^\dagger\gamma^0\gamma^\nu u(3)] = \bar{u}(1)\gamma^\nu u(3)$

- Completeness relations:

$$\sum_s u(\bar{p},s)\,\bar{u}(\bar{p},s) = \not{p}+m \qquad (5.79)$$

$$\sum_s v(\bar{p},s)\,\bar{v}(\bar{p},s) = \not{p}-m$$

- Because $a^\dagger b = \mathrm{Tr}(a^\dagger b) = \mathrm{Tr}(b a^\dagger)$ for any vectors a, b;

$$\sum_s{}' \bar{u}(\bar{p},s)\, \underset{\text{any } 4\times4\text{-matrix}}{M}\, u(\bar{p},s) = \sum_s{}' \mathrm{Tr}\left[M u(\bar{p},s)\bar{u}(\bar{p},s)\right]$$

$$= \mathrm{Tr}\left[M(\not{p}+m)\right] \qquad (5.80)$$

Thus,

$$\langle |\mathcal{M}|^2\rangle = \frac{e^2}{4(q_1-p_3)^2}\,\mathrm{Tr}\left[\gamma^\mu(\not{q}_1+m_e)\gamma^\nu(\not{p}_3+m_e)\right]\times$$

$$\mathrm{Tr}\left[\gamma_\mu(\not{p}_4-m_\mu)\gamma_\nu(\not{q}_2-m_\mu)\right] \qquad (5.81)$$

Using $\mathrm{Tr}[\gamma^\mu \not{a}\gamma^\nu \not{b}] = 4(a^\mu b^\nu + a^\nu b^\mu - g^{\mu\nu} a\cdot b)$ we obtain

$$\mathrm{Tr}\left[\gamma^\mu(\not{q}_1+m_e)\gamma^\nu(\not{p}_3+m_e)\right] = 4\left(q_1^\mu p_3^\nu + q_1^\nu p_3^\mu - g^{\mu\nu}(q_1\cdot p_3 - m_e^2)\right)$$

etc.

Inserting this into (5.41) or (5.45) we obtain the expression for cross-section.

5.8 Fermion sign

- Diagrams with fermions have sign ± 1, related to anticommutation (permutations)
- If we study only 1 diagram, this usually does not matter, but with more than 1 the relative sign is important.
- Rule: swap 2 fermions $\rightarrow$ change sign
- Consider $G^{(4)}(x_1, x_2, x_3, x_4) = \langle 0 | T(\psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4)) | 0 \rangle$
- Source terms $e^{i\int d^4x (\bar{J}\psi + \bar{\psi}J)} \equiv A$

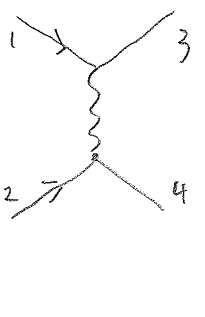
$$\Rightarrow \frac{\partial}{i\partial \bar{J}_x} A = \psi_x A \; ; \quad \frac{\partial}{i\partial J_x} A = -\bar{\psi}_x A \qquad (5.82)$$

$$\frac{\partial}{i\partial \bar{J}_x} \frac{\partial}{i\partial J_y} = -\psi_x \bar{\psi}_y A \qquad \text{etc.}$$

Thus, now $\underline{G^4(x_1,x_2,x_3,x_4) = \bar{\partial}_1 \bar{\partial}_2 \partial_3 \partial_4 Z[J]\big|_{J=0}}$ (5.83)

where $\bar{\partial}_1 \equiv \frac{\partial}{i\partial \bar{J}(x_1)}$; $\partial_2 \equiv \frac{\partial}{i\partial J(x_2)}$

- Consider now graph



The interaction lagrangian part

$$\mathcal{L}_I = \bar{\psi}\, ie \not{A} \psi$$

$$Z[J] = e^{\,i\int d^4x\, \mathcal{L}_I\left(\underset{\substack{\uparrow \\ \bar{\psi}_x}}{\frac{-\partial}{i\partial J_x}},\ \underset{\substack{\uparrow \\ \psi_x}}{\frac{\partial}{i\partial \bar{J}_x}}\right)} Z_0[J] \qquad (5.84)$$

- Thus, for the diagram at hand we need to expand e^{iS_I} to 2nd order to obtain 2 vertices.
- We now keep track only of the sign due to Grassmann-numbers. Denote

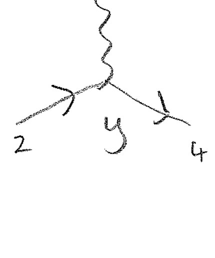
$$\left(\frac{\partial}{i\partial J_x}\, ie\not{A}\, \frac{-\partial}{i\partial \bar{J}_x}\right) \longrightarrow (\partial\bar{\partial})_x$$

- Thus, $\quad G^{(4)} = \bar{\partial}_1 \bar{\partial}_2 \partial_3 \partial_4 \int_x \int_y (\partial\bar{\partial})_x (\partial\bar{\partial})_y \, Z_0[J]\Big|_{J=0}$
- Now $Z_0[J] = e^{-\int d^4x\, d^4y\, \bar{J}_x S_F(x-y) J_y}$

Thus, application of $\partial_x Z_0[J] = +i\int dy\, \bar{J}_y S_F(y-x)\, Z_0[J]$

and we need to operate with $\bar{\partial}_y$ to get non-vanishing result : $\underbrace{\bar{\partial}_y \partial_x}_{\text{"Contraction"}} Z_0[J]\Big|_{J=0} = S_F(x-y)$

We need to pairwise contract all ∂'s:



$$\bar{\partial}_1 \bar{\partial}_2 \partial_3 \partial_4 (\partial\bar{\partial})_x (\partial\bar{\partial})_y Z_0$$

- Permute ∂'s so that you obtain

$$\bar\partial\partial\;\bar\partial\partial\;\ldots\;\bar\partial\partial\;Z_0 \longrightarrow (-1)^{\#\text{ of permutations}}$$

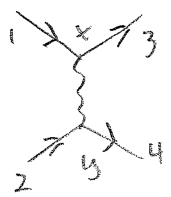
(∂'s are Grassmann-derivatives!) Easiest:

a) Swap $\partial, \bar\partial$ so that lines do not cross : $(-1)^{\text{swaps}}$

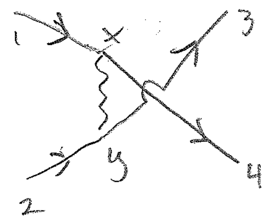
b) $\partial_a\bar\partial_b \rightarrow \bar\partial_b\partial_a$ gives (-1)

Above: swap $\bar\partial_1, \partial_4$ $\rightarrow$ lines do not cross (-1)

twice $\partial\bar\partial \rightarrow \bar\partial\partial$: $(-1)^2$



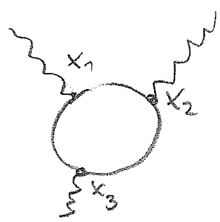
$$\bar\partial_1\bar\partial_2\partial_3\partial_4(\partial\bar\partial)_x(\partial\bar\partial)_y \rightarrow (-1)$$



$$\bar\partial_1\bar\partial_2\partial_3\partial_4(\partial\bar\partial)_x(\partial\bar\partial)_y \rightarrow (+1)$$

"swap fermions $\rightarrow$ different sign!"

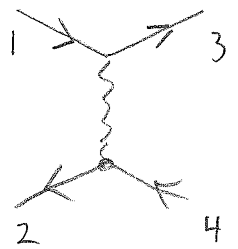
<u>Closed loop</u>: no external legs



$$(\partial\bar\partial)_{x_1}(\partial\bar\partial)_{x_2}(\partial\bar\partial)_{x_3}\ldots(\partial\bar\partial)_{x_n}$$

No crossings, but one $\partial\bar\partial \rightarrow \bar\partial\partial \Rightarrow -1$, or $-$trace.

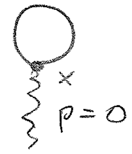
Antiparticle:



$$\bar{\partial}_1 \partial_2 \partial_3 \bar{\partial}_4 (\partial \bar{\partial})_x (\partial \bar{\partial})_y \rightarrow (+1)$$

Note:

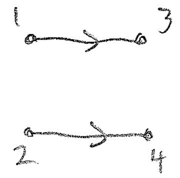
- Contraction $(\partial \bar{\partial})_x \longrightarrow$



does not appear!

- Contraction

$$\bar{\partial}_1 \bar{\partial}_2 \partial_3 \partial_4 \cdots \longrightarrow$$



disconnected, does not contribute to $\mathcal{M}$!