Propagators describe "propagation" of particle :

$$\langle 0|\hat{\varphi}(x)\,\hat{\varphi}(y)|0\rangle = \overset{y}{\bullet}\longrightarrow\overset{x}{\bullet}$$

$\hat{a}^+$ generates particle at point y

$\hat{a}$ annihilates it at x

But not quite! We can define more elaborate 2-pt. functions:

- $$\rho(x,y) \equiv \langle 0|\tfrac{1}{2}[\hat{\varphi}(x),\hat{\varphi}(y)]|0\rangle \qquad (1.33)$$

  spectral function

  $$\Delta(x,y) \equiv \langle 0|\tfrac{1}{2}\{\hat{\varphi}(x),\hat{\varphi}(y)\}|0\rangle$$

  here $\{A,B\} = AB+BA$

- $$G_R(x,y) \equiv i\,\Theta(x^0-y^0)\langle 0|[\hat{\varphi}(x),\hat{\varphi}(y)]|0\rangle \qquad (1.34)$$

  "Retarded Green function"

- $$G_A(x,y) \equiv -i\,\Theta(y^0-x^0)\langle 0|[\hat{\varphi}(x),\hat{\varphi}(y)]|0\rangle \qquad (1.35)$$

  "Advanced Green function"

Here $$\Theta(x) = \begin{cases} 1, & \text{if } x>0 \\ 0, & \text{if } x<0 \end{cases}$$

Heaviside step function

- $$G_F(x,y) \equiv \langle 0 | \Theta(x^0-y^0)\,\hat{\varphi}(x)\,\hat{\varphi}(y) + \Theta(y^0-x^0)\,\hat{\varphi}(y)\,\hat{\varphi}(x) | 0 \rangle \qquad (1.36)$$
$$\equiv \langle 0 | T\{\hat{\varphi}(x)\,\hat{\varphi}(y)\} | 0 \rangle$$

"Time-ordered Green function" or

<u>Feynman propagator</u>

Of these, retarded Green function and Feynman propagator are the most important

- Retarded propagator: physical propagation
- Feynman propagator: appears in perturbative expansions (Feynman diagrams)

Also useful is the "Euclidean Green function" or <u>Schwinger propagator</u>:

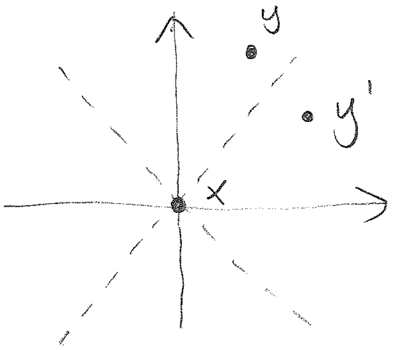
- $$G_E(\tilde{x},\tilde{y}) \equiv \langle 0 | \Theta(\tilde{x}^0-\tilde{y}^0)\,\hat{\varphi}(\tilde{x})\hat{\varphi}(\tilde{y}) + \Theta(\tilde{y}^0-\tilde{x}^0)\hat{\varphi}(\tilde{y})\,\hat{\varphi}(\tilde{x}) | 0 \rangle$$

where $\tilde{x}^0 = i x^0 \in \mathbb{R}$, $\tilde{y}^0 = i y^0 \in \mathbb{R}$ $\Rightarrow$ $x^0 = -i\tilde{x}^0$, $y^0 = -i\tilde{y}^0$ (1.37)

Thus, here we have imaginary time!

1.5 Causality and propagators

- Quantum field theories obey strict causality: events outside of light cone cannot affect each other:



$$\langle 0|[\hat{\varphi}(x), \hat{\varphi}(y)]|0\rangle \neq 0$$

$$\underline{\langle 0|[\hat{\varphi}(x), \hat{\varphi}(y')]|0\rangle = 0,}$$

if $(x-y')^2 < 0$, i.e. space-like difference!

- This is not satisfied in (relativistic) QM:

A) Non-relativistic QM: propagation $(x^0=0, \bar{x}) \to (y^0=t, \bar{y})$ described by the matrix element

$$\langle \bar{y}| e^{-i\hat{H}t} |\bar{x}\rangle \qquad \text{assume } \hat{H} = \frac{\hat{\bar{p}}^2}{2m}, \text{ free particle}$$

$$\uparrow \int d^3\bar{p}\, |\bar{p}\rangle\langle\bar{p}|$$

$$= \langle \bar{y}| \int d^3\bar{p}\, e^{-i\frac{\bar{p}^2}{2m}t} |\bar{p}\rangle\langle\bar{p}|\bar{x}\rangle \qquad \langle\bar{p}|\bar{x}\rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\bar{p}\cdot\bar{x}}$$

$$= \frac{1}{(2\pi)^3} \int d^3\bar{p}\, e^{-i\frac{\bar{p}^2}{2m}t - i\bar{p}\cdot(\bar{x}-\bar{y})}$$

$$= \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\bar{y}-\bar{x})^2/2t} \qquad (1.38)$$

$\neq 0$ for any $\bar{x}-\bar{y}$ and t. $\Rightarrow$ superluminal.

No wonder, $\hat{H}$ was non-relativistic.

B) How about $\hat{H} = \sqrt{\hat{p}^2 + m^2}$?

Now $\langle \bar{y} | e^{-it\sqrt{\hat{p}^2+m^2}} | \bar{x} \rangle =$

$$= \frac{1}{(2\pi)^3} \int d^3\bar{p}\, e^{-it\sqrt{p^2+m^2} - i\bar{p}\cdot(\bar{x}-\bar{y})}$$

$$= \frac{1}{(2\pi)^3 |\bar{x}-\bar{y}|} \int_0^\infty dp\, p \underbrace{\sin(p|\bar{x}-\bar{y}|)}_{\sim \exp(ip|\bar{x}-\bar{y}|)} e^{-it\sqrt{p^2+m^2}} \qquad (1.39)$$

The integrand oscillates wildly, but we can estimate it using stationary phase principle: the integral contributes when $\frac{\delta\,\text{phase}}{\delta p} \simeq 0$

$$\text{phase} = \pm p_0 |\bar{x}-\bar{y}| - t\sqrt{p_0^2 + m^2} = 0$$

$$\Rightarrow p_0^2 = \frac{m^2 |\bar{x}-\bar{y}|^2}{t^2 - |\bar{x}-\bar{y}|^2} \qquad \text{if } t^2 > |\bar{x}-\bar{y}|^2$$

Expand around p_0: $p' = p - p_0 \Rightarrow \ldots$

$\Rightarrow$

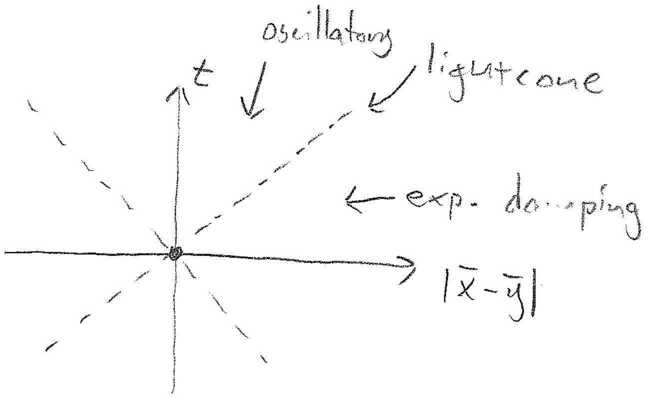
$$(1.39) \simeq e^{im\sqrt{t^2 - |\bar{x}-\bar{y}|^2}} \qquad (1.40)$$

$$\simeq e^{imt}, \quad t \to \infty, \quad |\bar{x}-\bar{y}| \text{ const.}$$

If $t^2 < |\bar{x}-\bar{y}|^2$, by analytic continuation

$$(1.39) \simeq e^{-m\sqrt{|\bar{x}-\bar{y}|^2 - t^2}} \qquad (1.41)$$

Thus, we obtain:



Better but not sufficient! There should be nothing outside the light cone!

- In QFT, the Wightman function $\langle 0|\hat{\varphi}(y)\,\hat{\varphi}(x)|0\rangle$ behaves as (1.40-41).

Thus, it is not the "proper" propagator

c) Let us look instead at

$$\rho(x,y) = \langle 0|\tfrac{1}{2}[\hat{\varphi}(x), \hat{\varphi}(y)]|0\rangle$$

$$= \frac{1}{2}\int_{\bar{p}}\int_{\bar{q}} \langle 0|[\hat{a}_{\bar{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\bar{p}} e^{ip\cdot x},\ \hat{a}_{\bar{q}} e^{-iq\cdot y} + \hat{a}^\dagger_{\bar{q}} e^{iq\cdot y}]|0\rangle$$

$$= \frac{1}{2}\int \frac{d^3\bar{p}}{(2\pi)^3\, 2E_{\bar{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)$$

$$= \frac{1}{2}\int \frac{d^3\bar{p}}{(2\pi)^3\, 2E_{\bar{p}}} \left(e^{-iE_{\bar{p}}(x^0-y^0) - i\bar{p}\cdot(\bar{x}-\bar{y})} - e^{iE_{\bar{p}}(x^0-y^0) + i\bar{p}\cdot(\bar{x}-\bar{y})}\right)$$

$\bar{p} \to -\bar{p}$ here

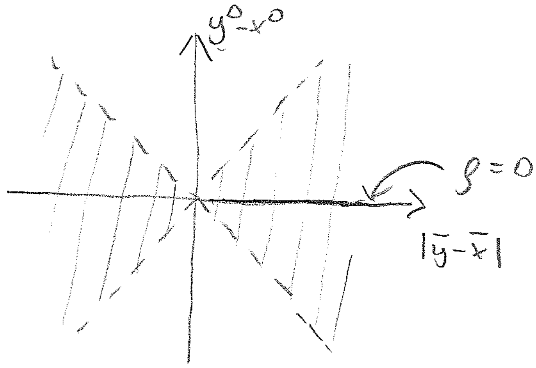
$$= \int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}} e^{-i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{1}{2}\left(e^{-iE_{\bar{p}}(x^0-y^0)} - e^{iE_{\bar{p}}(x^0-y^0)}\right) \tag{1.42}$$

This is like (1.39), except we have also the negative energy part. This has the following consequences:

* $\underline{\rho(x,y) = 0}$ if $\underline{x^0 = y^0}$ (1.43)

* $\frac{\partial}{\partial y^0}\rho(x,y)\Big|_{x^0=y^0} = \frac{i}{2}\int \frac{d^3\bar{p}}{(2\pi)^3} e^{-i\bar{p}\cdot(\bar{x}-\bar{y})} = \frac{i}{2}\delta^{(3)}(\bar{x}-\bar{y})$

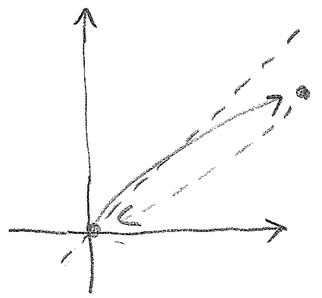
* (1.43) implies that $\rho(x,y) = 0$ in an environment around $x^0 = y^0$ where $\rho(x,y)$ is analytic.
This domain is the whole <u>spacelike</u> separation $(x-y)^2 \equiv (x^0-y^0)^2 - (\bar{x}-\bar{y})^2 < 0$!



$\rho = 0$ along $x^0 = y^0$ $\Rightarrow$ $\rho = 0$ if $(x-y)^2 < 0$!

* For timelike $(x-y)^2 > 0$ $\rho(x,y) \neq 0$; if $\bar{x} = \bar{y}$, $\rho(x,y) \sim e^{im(y^0-x^0)}$

* For this to be true it was necessary to have "negative energy" branch:



propagation backwards in time; antiparticle forward in time!

* The commutator $[\hat{\varphi}(x), \hat{\varphi}(y)]$ tells if points x, y are causally connected. For complex fields $[\hat{\phi}(x), \hat{\phi}^{+}(y)]$

(Note: $\langle 0 | [\hat{\varphi}(x), \hat{\varphi}(y)] | 0 \rangle = [\hat{\varphi}(x), \hat{\varphi}(y)]$

but $\underbrace{\langle 0 | \hat{\varphi}(x) \hat{\varphi}(y) | 0 \rangle}_{\text{number}} \neq \underbrace{\hat{\varphi}(x) \hat{\varphi}(y)}_{\text{operator}}$)

Feynman propagator:

Because $\langle 0|\hat{\varphi}(x)\,\hat{\varphi}(y)|0\rangle =$

$$= \int_{\bar{p}}\int_{\bar{q}} \langle 0|\left(\hat{a}_{\bar{p}} e^{-ip\cdot x} + \hat{a}^{\dagger}_{\bar{p}} e^{ip\cdot x}\right)\left(\hat{a}_{\bar{q}} e^{-iq\cdot y} + \hat{a}^{\dagger}_{\bar{q}} e^{iq\cdot y}\right)|0\rangle$$

$$[\hat{a}_{\bar{p}},\hat{a}^{\dagger}_{\bar{q}}] + \hat{a}^{\dagger}_{q}\hat{a}_{p}$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}}\, e^{-ip\cdot(x-y)}$$

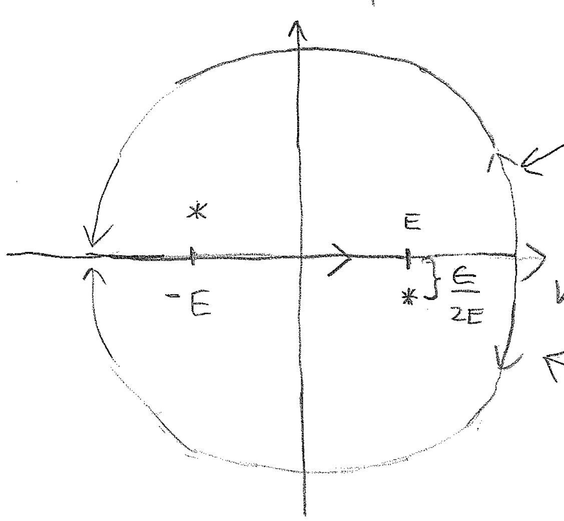
$$\Rightarrow\ G_F(x,y) \equiv \langle 0|T\{\hat{\varphi}(x)\,\hat{\varphi}(y)\}|0\rangle$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}}\left[\Theta(x^0-y^0)\, e^{iE_{\bar{p}}(y^0-x^0)} + \Theta(y^0-x^0)\, e^{iE_{p}(x^0-y^0)}\right] \times e^{-i\bar{p}\cdot(\bar{y}-\bar{x})}$$

Very useful relation: (1.44)

$$\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{e^{ik\tau}}{k^2-E^2+i\epsilon} = -\frac{i}{2E}\left[\Theta(-\tau)\, e^{iE\tau} + \Theta(\tau)\, e^{-iE\tau}\right]$$

poles at $k = \pm\left(E - i\frac{1}{2E}\epsilon\right)$



$\tau > 0$, $2\pi i \times \mathrm{Res} = -\frac{i}{2E} e^{iE\tau}$

$\tau < 0$, $-2\pi i \times \mathrm{Res} = -\frac{i}{2E} e^{-iE\tau}$

Thus,

$$G_F(x,y) = i\int \frac{d^3\bar{p}}{(2\pi)^3}\, e^{-i\bar{p}\cdot(\bar{y}-\bar{x})} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{e^{ik(y^0-x^0)}}{k^2 - E_{\bar{p}}^2 + i\epsilon} \Rightarrow$$

$$\boxed{G_F(x,y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot(x-y)}} \qquad (1.45)$$

Here $p^2 = p^{0\,2} - \bar{p}^2$. Relation $p^0 = E_{\bar{p}} \equiv \sqrt{\bar{p}^2 + m^2}$ not there!

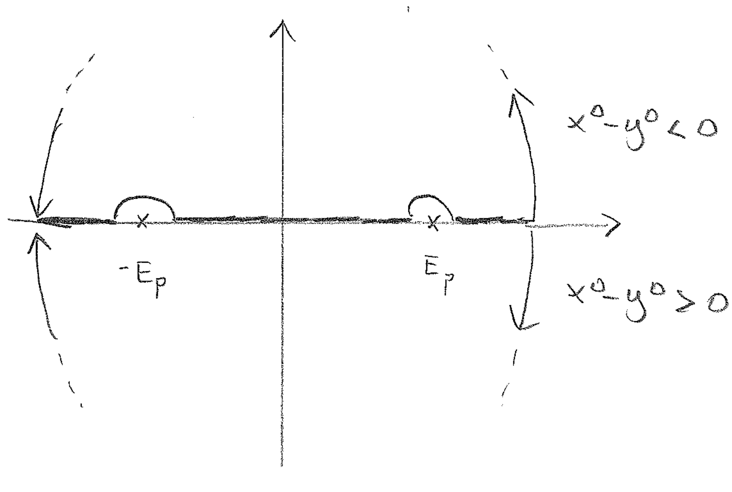
<u>Retarded propagator</u>

$$G_R(x,y) = 2\,\Theta(x^0 - y^0)\,\rho(x,y) \qquad (1.42) \Rightarrow$$

$$= \Theta(x^0-y^0)\int \frac{d^3\bar{p}}{(2\pi)^3}\left[\frac{e^{-ip\cdot(x-y)}}{2E_{\bar{p}}} + \frac{e^{ip\cdot(x-y)}}{-2E_{\bar{p}}}\right] \qquad p^0 = E_{\bar{p}}$$

$$= \boxed{\int \frac{d^3\bar{p}}{(2\pi)^3} \int_C \frac{dp^0}{2\pi}\, \frac{i}{p^2 - m^2}\, e^{-ip\cdot(x-y)}} \qquad (1.46)$$

here $p^2 = p^{0\,2} - \bar{p}^2$, and p^0-integral is along C:



$$p^2 - m^2 = p^{0\,2} - \bar{p}^2 - m^2$$
$$= (p^0 + E_p)(p^0 - E_p)$$

If $x^0 - y^0 < 0$, $\int_C dp^0 = 0$,

if $x^0 - y^0 > 0$, $\int_C dp^0 = -2\pi i \times (\text{Residues})$

To summarize,

$$G_{F,R}(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)} \qquad (1.45)$$

where p^0 - contour is

Feynman　　　　Retarded

These are <u>Green functions</u>, i.e. obey

$$[\partial^\mu_x \partial_{x\mu} + m^2]\, G = -i\, \delta^{(4)}(x-y) \qquad (1.46)$$

$[\partial^\mu_x \partial_{x\mu} + m^2]\, G_F(x,y)$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(-p^{0\,2} + \vec{p}^{\,2} + m^2)\, e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4} (-i)\, e^{-ip\cdot(x-y)} = -i\, \delta^{(4)}(x-y). \; \square$$

Same for G_R.

All of the previous was with free (= non-interacting) Klein-Gordon fields. In interacting case we have all kinds of Green functions:

$$G_T = \langle 0 | T\{\hat{\varphi}(x_1)\hat{\varphi}(x_2)\dots\hat{\varphi}(x_n)\}|0\rangle$$

These can be calculated in canonical quantisation as we have discussed, but we shall discuss these in terms of path integral quantisation.