2. Path integrals and scalar fields

Path integral method is the most general way to treat QFT's. It is explicitly Lorentz-invariant, and respects all symmetries of the original theory.

2.1. Quantum mechanics and P. I.

(Feynmann, Hibbs ; Schwinger)

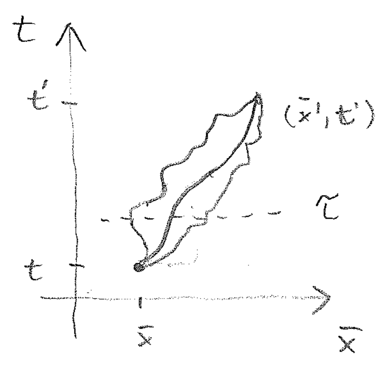
Let us consider the amplitude

$$\langle \bar{x}(t') | \bar{x}(t) \rangle_S = \langle \bar{x}' | e^{-i\hat{H}(t'-t)} | \bar{x} \rangle_H$$

In path integral, this becomes

$$= \int \Big[\prod_{\substack{\tau \\ t<\tau<t'}} dx(\tau) \Big] e^{\frac{i}{\hbar} S[\bar{x}]} \qquad (2.1)$$

This is a sum over all paths $\bar{x}(t) \rightarrow \bar{x}'(t')$, weighted by $e^{iS/\hbar}$:



One of the paths is the class. solution with extremal S

Let us show this:

Now $\langle \bar{x} | \bar{p} \rangle = e^{i\bar{p}\cdot\bar{x}} = \langle \bar{p} | \bar{x} \rangle^*$

$$\int d^3\bar{x}\, |\bar{x}\rangle\langle\bar{x}| = \mathbb{1}$$

$$\int \frac{d^3\bar{p}}{(2\pi)^3} |\bar{p}\rangle\langle\bar{p}| = \mathbb{1} \qquad (2.2)$$

$$\left[\ \mathbb{1} = \int d^3\bar{x} \int d^3\bar{x}' \int \frac{d^3\bar{p}}{(2\pi)^3} |\bar{x}\rangle \overbrace{\langle\bar{x}|\bar{p}\rangle\langle\bar{p}|\bar{x}'\rangle}^{e^{i\bar{p}\cdot(\bar{x}-\bar{x}')}} \langle\bar{x}'| \right.$$

$$\left. = \int d^3\bar{x}\, |\bar{x}\rangle\langle\bar{x}| = \mathbb{1} \ ; \ \text{Thus, normalization OK} \ \right]$$

Hamilton $\hat{H} = \frac{\hat{p}}{2m} + V(\hat{x})$

Infinitesimal step:

$$\langle \bar{x}'(t+\delta t) | \bar{x}(t) \rangle = \langle \bar{x}' | e^{-i\hat{H}\delta t} | \bar{x} \rangle$$

$$= \langle \bar{x}' | (1 - i\delta t \hat{H}) | \bar{x} \rangle + \mathcal{O}(\delta t^2) \qquad (**)$$

$$= \langle \bar{x}' | \int \frac{d^3\bar{p}}{(2\pi)^3} |\bar{p}\rangle\langle\bar{p}| \left(1 - i\delta t\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]\right) |\bar{x}\rangle = \ldots$$

Reminder:

$$\begin{cases} f(\hat{x})|\bar{x}\rangle = f(\bar{x})|\bar{x}\rangle \\ \langle a|\hat{O} = \langle \hat{O}^{+} a| \ ; \quad \hat{p}^{+} = \hat{p}, \ \hat{x}^{+} = \hat{x} \\ \Rightarrow \langle\bar{p}|f(\hat{p}) = \langle\bar{p}|f(\bar{p}) \end{cases}$$

$$\ldots = \int \frac{d^3\bar{p}}{(2\pi)^3} \left(1 - i\delta t\left[\frac{\bar{p}^2}{2m} + V(\bar{x})\right]\right) e^{i\bar{p}\cdot(\bar{x}'-\bar{x})}$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3} \exp\left[i\bar{p}\cdot(\bar{x}'-\bar{x}) - i\,\delta t\left(\frac{\bar{p}^2}{2m} + V(\bar{x})\right)\right] + \mathcal{O}(\delta t^2)$$

Substitute $\bar{x}' - \bar{x} = \delta t \cdot \dot{\bar{x}} + \mathcal{O}(\delta t^2)$

The integral is Gaussian! Write exponent as

$$-i\frac{\delta t}{2m}(\bar{p} - m\dot{\bar{x}})^2 + i\frac{\delta t}{2}m\dot{\bar{x}}^2 - i\,\delta t\,V(\bar{x})$$

Integral over $\bar{p}' = \bar{p} - m\dot{\bar{x}}$ is straightforward, with

$$\ldots = \frac{1}{(2\pi)^3}\left[\frac{2m\pi}{i\,\delta t}\right]^{3/2} \exp\Big[i\,\delta t\underbrace{\left(\tfrac{1}{2}m\dot{\bar{x}}^2 - V(\bar{x})\right)}_{=L=T-V=\dot{\bar{x}}\cdot\bar{p}-H}\Big] \qquad (2.3)$$

$$= \left[\frac{m}{i\pi\,\delta t}\right]^{3/2} \exp\Big[i \underbrace{\int_t^{t+\delta t} dt'\, L}_{S_{t,t+\delta t}}\Big]$$

With $\hbar$ explicit, we obtain $\left[\frac{m\hbar}{i\pi\,\delta t}\right]^{3/2} e^{\frac{i}{\hbar}S}$

$\hbar \to 0 \Rightarrow$ only extremal of S contributes

$\Rightarrow$ classical solution

NOTE: expansion $e^{-i\hat{H}\delta t} = 1 - i\hat{H}\delta t$

can be avoided, using Baker-Campbell-Hausdorff formula. Then no need to re-exponentiate, But the above is simpler.

Finite time interval: divide it to N,
let $N \to \infty$

$$\langle \bar{x}'(t') | \bar{x}(t) \rangle = \int \Big[\prod_{i=1}^{N-1} dx_i \Big] \langle \bar{x}'(t') | x_{N-1} \rangle \langle x_{N-1} | x_{N-2} \rangle \times \ldots$$

$$\ldots \langle x_2 | x_1 \rangle \langle x_1 | x(t) \rangle$$

with $|x_i\rangle \equiv |x(t_i)\rangle$, $t_i = t + \delta t \cdot i$, $\delta t = \frac{t-t'}{N}$

Using (2.3) to each of $\langle x_{i+1} | x_i \rangle$,

$$\langle \bar{x}'(t') | \bar{x}(t) \rangle = \int \Big[\prod_{i=1}^{N-1} dx_i \Big] \left(\frac{m}{i\pi \delta t} \right)^{3N/2} e^{\underbrace{i \sum_i \delta t}_{\int_t^{t'} dt} \left(\frac{1}{2} m \dot{\bar{x}}_i^2 - V(\bar{x}_i) \right)}$$

$$= \underline{\int Dx \, \exp\Big[i \int_t^{t'} d\tau \, L(x(\tau)) \Big]} \qquad (2.4)$$

when $N \to \infty$ ($\delta t \to 0$). This is the desired result.

* Normalization in front (usually) not important, and we set it to 1.

* Boundary conditions fixed: at t, t' x fixed

* Matrix element in terms of purely "classical" terms: no operators, "states" etc!

* Beautiful applications to std. QM processes in book by Feynman and Hibbs!

* However, this method is not used much for QM problems — Schrödinger eqn. simpler!

2.2. Functional methods etc.

Here we shall describe useful mathematical tools for path integrals & field theory

Gaussian integral:

$$F[A,\omega] = \int \left[\prod_{i=1}^{N} dx_i\right] e^{-x^T A x + \omega^T x} = \pi^{N/2} (\det A)^{-1/2} e^{\frac{1}{4}\omega^T A^{-1} \omega} \qquad (2.5)$$

A is $N\times N$ real and symmetric matrix (only symm. part relevant!)
ω, x N-comp. real vectors

Proof: $x^T A x - \omega^T x = (x - \frac{1}{2}A^{-1}\omega)^T A (x - \frac{1}{2}A^{-1}\omega) - \frac{1}{4}\omega^T A^{-1}\omega$

- Let $x' = x - \frac{1}{2}A^{-1}\omega$

$$\prod_i dx_i' = \left\| \frac{dx'}{dx} \right\| \prod_i dx_i = \prod_i dx_i$$

$$\Rightarrow F[A,\omega] = e^{-\frac{1}{4}\omega^T A^{-1}\omega} \int [\prod_i dx_i'] \, e^{-x'^T A x'}$$

- A symmetric, $A^T = A \Rightarrow$ can diagonalize:

$$A_D = \mathrm{diag}(\lambda_1, \lambda_2, \dots \lambda_N) = O^T A O \quad , \quad O^T O = \mathbb{1}$$

orthogonal

- Let $y = O^T x'$; $y^T = x'^T O$

$$\prod_i dy_i = \left\| \frac{dy}{dx} \right\| \prod_i dx_i = |\det O^T| \prod_i dx_i = \prod_i dx_i$$

$$\Rightarrow \int [\prod_i dx_i'] \, e^{-x'^T A x'} = \int [\prod_i dy_i] \, e^{-y^T A_D y}$$

$$= \prod_i \int dy \, e^{-\lambda_i y^2} = \prod_i \sqrt{\frac{\pi}{\lambda_i}} \qquad (2.6)$$

- $\det A = \prod_i \lambda_i \quad \Rightarrow \quad \square$.

- Convergence requires $\lambda_i > 0 \;\; \forall i$.

Can be generalized to non-real A, as long as $\mathrm{Re}\, \lambda_i \geq 0$, $\lambda_i \neq 0 \;\; \forall i$.

- In QFT, λ_i often pure imaginary

Complex version:

$$\int \left[\prod_{i=1}^{N} dz\,dz^*\right] e^{-z^\dagger A z + w^\dagger z + z^\dagger w}$$

$A^\dagger = A$ Hermitean ; z, w complex vectors

$$= \pi^N (\det A)^{-1} e^{-w^\dagger A^{-1} w} \qquad (2.6)$$

(product of 2 real integrals, for $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$)

Functionals:

Functions which have a function as an argument. Examples

$$F_1[f] = \int_a^b dx\, f(x) \qquad \text{number } (\mathbb{R}, \mathbb{C})$$

$$F_2[f](x) = f^2(x) - f(x) \qquad \text{function}$$

Functional derivative is defined through

$$\boxed{\frac{\delta F[f]}{\delta f(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ F[f(x) + \epsilon\, \delta(x-y)] - F[f(x)] \right\}} \qquad (2.7)$$

Thus,

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y) \qquad \text{Useful!} \qquad (2.8)$$

$$\frac{\delta F_1[f]}{\delta f(y)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_a^b dx\, \epsilon\, \delta(x-y) = \begin{cases} 1, & \text{if } a<y<b \\ 1/2, & \text{if } y=a \text{ or } y=b \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{\delta F_2[f](x)}{\delta f(y)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \Big\{ (f(x)+\epsilon\,\delta(x-y))^2 - f(x) - \epsilon\,\delta(x-y) - f(x)^2 + f(x) \Big\}$$
$$= (2f(x)-1)\;\delta(x-y)$$

Works as expected!

* <u>Simple rule</u>:

$$\frac{\delta F[f]}{\delta f(y)} \quad \text{works as formal derivative wrt. } f \text{, with (2.8) stuck at end!} \qquad (2.9)$$

- Example: $E[f] = \int dx\, dy\, K(x,y)\, f(x)\, f(y)$ "Kernel"

$$\frac{\delta E}{\delta f(z)} = \int dy\, K(z,y)\, f(y) + \int dx\, K(x,z)\, f(x)$$

$$\frac{\delta^2 E}{\delta f(z_1)\, \delta f(z_2)} = K(z_2, z_1) + K(z_1, z_2) \qquad (2.10)$$

If $G[f] = \int dx_1 .. dx_n \, K(x_1, .., x_n) \, f(x_1) .. f(x_n)$

with K symmetric in permutations of x_i,

we obtain

$$\frac{\delta^n G[f]}{\delta f(y_1) \dots \delta f(y_n)} = n! \, K(y_1, y_2, .. y_n) \qquad (2.11)$$

2.3. Path integral for scalar fields

In analogy with path integral in QM, we obtain path integral representation for field theory. This is perhaps the most general way to formulate the theory:

Feynman path integral (1948)

$$\boxed{Z = \int D\varphi \, e^{iS} \, , \quad S = \int d^4x \, \mathcal{L}} \qquad (2.12)$$

defines the theory! Here $D\varphi = \prod_x d\varphi(x)$, independent integration variable at each space-time location

Observables:

Feynman propagator

$$G_F(x-y) = \langle 0 | T \hat{\varphi}(x) \hat{\varphi}(y) | 0 \rangle = \frac{1}{Z} \int D\varphi \, \varphi(x) \varphi(y) \, e^{iS} \qquad (2.13)$$

In general,

$$\langle 0 | T \{ f(\hat{\varphi}) \} | 0 \rangle = \frac{1}{Z} \int D\varphi \, f(\varphi) \, e^{iS} \qquad (2.14)$$

(2.12) is often used to define QFT.
It has several advantages over canonical formalism

- Lorentz invariance explicit, respects symmetries of $\mathcal{L}$
- No reliance on definitions of "states", "operators" etc. QM constructs, which may be ill-defined in some cases
- Only contains standard integrals
- Non-perturbative approaches, simulation

The proof of the equivalence is analogous to the QM case: consider $(\bar{x} \to \varphi(\bar{x}))$

$$\langle \varphi_b, t | \varphi_a, t=0 \rangle = \langle \varphi_b | e^{-i\hat{H}t} | \varphi_a \rangle$$

Using the generalizations

$$\mathbb{I} = \int \left[\prod_{\bar{x}} d\varphi(\bar{x})\right] |\varphi\rangle\langle\varphi| = \int \left[\prod_{\bar{x}} \frac{d\pi(\bar{x})}{2\pi}\right] |\pi\rangle\langle\pi|$$

$$\langle \varphi | \varphi' \rangle = \prod_{\bar{x}} \delta(\varphi(\bar{x}) - \varphi'(\bar{x})) \qquad (2.15)$$

$$\langle \varphi | \pi \rangle = \prod_{\bar{x}} e^{i\varphi(\bar{x})\pi(\bar{x})} = e^{i\int d^3\bar{x}\, \varphi(\bar{x})\pi(\bar{x})}$$

(Formally $|\varphi\rangle = \prod_{\bar{x}} |\varphi(\bar{x})\rangle$

- we can again divide t-interval in N slices : $t = N\,\delta t$

$$\langle \varphi_b | e^{-i\hat{H}t} | \varphi_a \rangle = \int \prod_{i=1}^{N-1} d\varphi_i \, \langle \varphi_a | e^{-i\hat{H}\delta t} | \varphi_{N-1} \rangle \times \ldots$$

$$\ldots \langle \varphi_2 | e^{-i\hat{H}\delta t} | \varphi_1 \rangle \langle \varphi_1 | e^{-i\hat{H}\delta t} | \varphi_a \rangle$$

Here $d\varphi_i \equiv \prod_{\bar{x}} d\varphi_i(\bar{x})$

- Squeeze in $\mathbb{1} = \int \left[\prod_x \frac{d\pi(\bar{x})}{2\pi} \right] |\pi\rangle\langle\pi| \equiv \int d\pi |\pi\rangle\langle\pi|$

as required.

Now $\langle \varphi | e^{-i\hat{H}\delta t} | \pi \rangle = e^{-iH(\varphi,\pi)\delta t} \langle \varphi | \pi \rangle + \mathcal{O}(\delta t^2)$

$$= e^{i\delta t \int d^3\bar{x} \underbrace{(\varphi\pi - \mathcal{H})}_{\mathcal{L}}} + \mathcal{O}(\delta t^2)$$

which is valid for all sensible $\hat{H}$ (which is a function of $\hat{\varphi}, \hat{\pi}$)

- Thus,

$$\langle \varphi_b | e^{-i\hat{H}t} | \varphi_a \rangle = \int \prod_{i=1}^{N-1} \left[\prod_{\bar{x}} d\varphi_i(\bar{x}) \right] e^{i \sum_{i=1}^{N} \delta t \int d^3\bar{x} \, \mathcal{L}(\varphi_i)}$$

$$= \int \left[\prod_x d\varphi(x) \right] e^{i \int d^4x \, \mathcal{L}(\varphi)} \qquad (2.16)$$

where $\varphi(x) = \varphi(\delta t \cdot i, \bar{x}) = \varphi_i(\bar{x})$

and <u>boundary conditions</u> $\varphi(0,\bar{x}) = \varphi_a(\bar{x})$ and $\varphi(t,\bar{x}) = \varphi_b(\bar{x})$ are fixed.

We can also show that

$$\langle 0 | T\{\hat{\varphi}(x)\,\hat{\varphi}(y)\} | 0 \rangle = \frac{\int D\varphi\ \varphi(x)\varphi(y)\, e^{iS}}{\int D\varphi\ e^{iS}} \qquad (2.17)$$

std. Heisenberg pict.

Here $\langle 0 | \ldots | 0 \rangle \to t = -\infty \ldots \infty$ in S

(Proof: use $\langle \varphi_a | e^{-i\hat{H}(t_a - x^0)}\, \hat{\varphi}_S(\bar{x})\, e^{-i\hat{H}(x^0 - y^0)}\, \hat{\varphi}_S(\bar{y})\, e^{-i\hat{H}(y^0 - t_b)} | \varphi_b \rangle$

divide in δt-intervals etc.

Here $\hat{\varphi}_S(\bar{x}) = e^{i\hat{H}t}\, \hat{\varphi}(t,\bar{x})\, e^{-i\hat{H}t}$; $\hat{\varphi}_S(\bar{x}) | \varphi_a \rangle = \varphi_a(x) | \varphi_a \rangle$)

Source field

Handy tool for path integral evaluation:

$$Z[J] = \int D\varphi\ e^{iS[\varphi] + i\int d^4x\, J(x)\varphi(x)} \qquad (2.18)$$

Here $J(x)$ is arbitrary, external field. This is introduced for the purpose of evaluating Green functions

$$\frac{\delta}{i\delta J(x)} \frac{\delta}{i\delta J(y)} \frac{F[J]}{F[0]}\Bigg|_{J=0} = \frac{\int D\varphi\, \varphi(x)\varphi(y) e^{iS}}{\int D\varphi\, e^{iS}} \qquad (2.19)$$

after δ's!

$$= \langle 0 | T\, \hat{\varphi}(x) \hat{\varphi}(y) | 0 \rangle$$

Here $\frac{\delta}{\delta J(x)} \int d^4y\, f(y) J(y) = f(x)$; $\quad \frac{\delta J(x)}{\delta J(y)} = \delta^{(4)}(x-y)$

Generalizes immediately to higher point functions (just add $\frac{\delta}{\delta J(x_i)}$'s)

* This for scalar field and J.

For general vector $\varphi^a(x) J^a(x) = \varphi^T J$,

for complex field. $J^* \phi + \phi^* J$

$$= 2 (J_R \phi_R + J_I \phi_I)$$

(we need this combination in order for $\mathcal{L}$ and S to be real)