NOTE : For a free theory the integral in (2.18) is of gaussian form (2.5):

Consider Klein-Gordon scalar

$$S = \int d^4x \left[\tfrac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \tfrac{1}{2} m^2 \varphi^2 \right] \qquad (2.20)$$

$$= \int d^4x \Big[-\tfrac{1}{2} \varphi \underbrace{(\underbrace{\partial_\mu \partial^\mu}_{\equiv \Box} + m^2)}_{= A \text{ in } (2.5)} \varphi \Big] + \underbrace{\text{surface term}}_{=0, \text{ we assume}}$$

Thus,

$$e^{W[J]} \equiv Z[J] = \int \mathcal{D}\varphi \, e^{i\int d^4x \left(-\frac{1}{2}\varphi(\Box + m^2)\varphi + J\varphi\right)}$$

$$= \text{Const.} \det(\Box + m^2) \, e^{+i\frac{1}{2}\int d^4x\, d^4y\, J(x)(\Box+m^2)^{-1} J(y)} \qquad (2.21)$$

How is this even defined?

- Discretize space : $\int d^4x \to \sum_x (\delta x)^4$

  $\partial_\mu^2 \varphi = \frac{1}{(\delta x)^2}\left(\varphi(x-1) - 2\varphi(x) + \varphi(x+1)\right)$

  $\Rightarrow A_{x,y}$ matrix with non-zero elements only if $x = y$ or $x = y \pm \hat{e}_\mu$ for some μ.

- Limit $\delta x \to 0$

Thus, $e^{W[0]} = Z[0] = \text{Const.}\det(\Box + m^2)$

and

$$\frac{Z[J]}{Z[0]} = e^{+i\frac{1}{2}\int d^4x\, d^4y\, J(x)(\Box+m^2)^{-1}J(y)} \qquad (2.22)$$

This is still somewhat formal; let us look at the integral (2.21) closer:

a) Convergence: note that $\Box + m^2$ Hermitean, i.e. real eigenvalues. Ensure convergence by giving the exp. a small real part:

$$(-\Box - m^2) \longrightarrow (-\Box - m^2 + i\epsilon) \qquad (2.23)$$

b) What is $A^{-1} = (\Box + m^2 - i\epsilon)^{-1}$?

From (1.46)

$$(\Box + m^2 - i\epsilon)\, G_F(x-y) = -i\,\delta^{(4)}(x-y)$$

$$\Rightarrow \quad \underline{G_F(x-y) = -i\,(\Box + m^2 - i\epsilon)^{-1}} \qquad (2.24)$$

$$(1.45) \Rightarrow \quad = \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot(x-y)}$$

Feynman propagator

Thus, $$\boxed{\frac{Z[J]}{Z[0]} = e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x)\, G_F(x-y)\, J(y)}} \qquad (2.25)$$

and, for example,

$$\underline{\langle \varphi(x)\, \varphi(y) \rangle} \equiv \langle 0 | T\{\hat{\varphi}(x)\, \hat{\varphi}(y)\} | 0 \rangle$$

$$= \frac{\delta}{i\,\delta J(x)} \frac{\delta}{i\,\delta J(y)} \frac{Z[J]}{Z[0]} \Bigg|_{J=0}$$

$$= \frac{\delta}{i\,i\,\delta J(x)} \left[\left(-\int d^4x'\, G_F(y-x')\, J(x') \right) \frac{Z[J]}{Z[0]} \right]_{J=0}$$

$$= \underline{G_F(x-y)} \qquad \left(G_F(x-y) = G_F(y-x) \right) \qquad (2.26)$$

OK! This was the definition of G_F ...

Diagrammatically this is denoted by

$\langle \varphi(x)\, \varphi(y) \rangle \; =$ 

Often we can <u>normalize $Z[0] = 1$</u>, if the normalization is not important

Homework: show that

$$\frac{\delta}{i\,\delta J(x_1)} \frac{\delta}{i\,\delta J(x_2)} \frac{\delta}{i\,\delta J(x_3)} Z[J] \Bigg|_{J=0} = 0 \qquad (2.27)$$

Holds actually for all odd number of points (for free theory!)

On the other hand,

$$\langle \varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4)\rangle \equiv \langle 0|T\{\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_4)\}|0\rangle$$

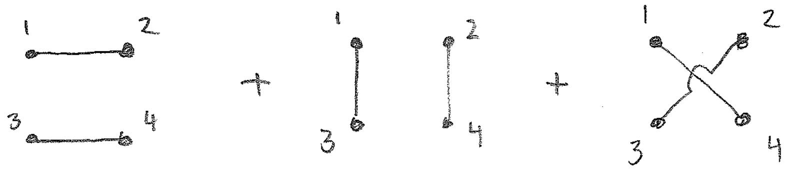
$$\frac{\delta^4}{i^4\,\delta J_1\cdots\delta J_4} Z[J]\Big|_{J=0} =$$

$$\frac{\delta^3}{\delta J_1\cdots\delta J_3}\Big|_{J=0}\left[-\int d^4x'\,G_F(x_4-x')\,J(x')\right] Z[J] =$$

one of $\frac{\delta}{\delta J_1}, \cdots \frac{\delta}{\delta J_3}$ must hit this, otherwise we get 0

$$= \underline{G_{12}\,G_{34} + G_{13}\,G_{24} + G_{14}\,G_{23}} \qquad (2.28)$$

$(G_{12} \equiv G_F(x_1-x_2)$ etc.$)$

$=$ 

In general, $\langle\varphi(x_1)\cdots\varphi(x_N)\rangle =$

$$\frac{\delta}{i\,\delta J(x_1)}\cdots\frac{\delta}{i\,\delta J(x_N)} Z[J]\Big|_{J=0} =$$ free theory

$$\sum_{\substack{\text{all}\\ \text{combinations}}} G_{12}\,G_{34}\,G_{56}\cdots G_{(N-1)N}$$

We say that $Z[J]$ generates Green's functions of the theory. What we have here is not very interesting because the theory is free. This generalizes to interacting theory too. (Soon...)

In Fourier space

Let us define Fourier transformations

$$\tilde{\varphi}(p) = \int d^4x \, \varphi(x) \, e^{ip\cdot x} \qquad (2.29)$$

$$\varphi(x) = \int \frac{d^4p}{(2\pi)^4} \tilde{\varphi}(p) e^{-ip\cdot x}$$

Now $S = \int d^4x \left(-\frac{1}{2}\varphi(x)(\partial_\nu \partial^\nu + m^2)\varphi(x) \right)$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{2} \tilde{\varphi}(-p)[p^2 - m^2] \tilde{\varphi}(p)$$

Because $\varphi(x) \in \mathbb{R}$, $\tilde{\varphi}(-p) = \tilde{\varphi}^*(p)$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{2} [p^2 - m^2] |\tilde{\varphi}(p)|^2 \qquad (2.30)$$

Now $\int \prod_x d\varphi(x) \, e^{iS[\varphi]} = \int \prod_p d\tilde{\varphi}(p) \, e^{iS[\tilde{\varphi}]}$

and

$$\langle \tilde{\varphi}^*(p) \tilde{\varphi}(q) \rangle \equiv \frac{\int D\tilde{\varphi} \, \tilde{\varphi}^*(p) \tilde{\varphi}(q) \, e^{iS}}{\int D\tilde{\varphi} \, e^{iS}} \qquad (2.31)$$

contains only simple gaussian integrals!

If $p \neq q$, integrand is odd w.r.t. $\varphi(q) \to -\varphi(q)$ and $\langle \tilde{\varphi}^*(p)\, \varphi(q) \rangle = 0$. Thus,

$$\underline{\underline{\langle \tilde{\varphi}^*(p)\, \tilde{\varphi}(q) \rangle}} = \delta^{(4)}(p-q) \frac{\int d\varphi\, d\varphi^*\ \varphi^* \varphi\, e^{i\frac{1}{2}[p^2-m^2]\varphi^*\varphi}}{\int d\varphi\, d\varphi^*\ e^{i\frac{1}{2}[p^2-m^2]\varphi^*\varphi}}$$

$$= \delta^{(4)}(p-q) \frac{i}{p^2-m^2}$$

From (1.45)

$$= \underline{\underline{\delta^{(4)}(p-q)\ \tilde{G}_F(p)}} \quad ! \qquad (2.32)$$

(For convergence, $[p^2-m^2] \to [p^2-m^2+i\epsilon]$ above!
$\Rightarrow$ Feynman prop.)

2.4. Interacting scalar field

Let us consider scalar field lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{1}{4!}\lambda \varphi^4 \qquad (2.33)$$
$$= \mathcal{L}_0 + \mathcal{L}_I$$

Now the action

$$S = \int d^4x\, \mathcal{L} = S_0 + S_I ,$$

$$\underline{S_I = -\int d^4x \frac{1}{4!}\lambda \varphi^4}$$

is <u>not</u> quadratic ($\varphi A \varphi$) in field φ, and we <u>cannot solve</u> the theory exactly.

What to do?

- S_I small perturbation (formally)
  $\Rightarrow$ <u>perturbation theory</u>
  $\Rightarrow$ power series in λ
- Numerical integration of $Z = \int \mathcal{D}\varphi\, e^{iS}$
  $\Rightarrow$ lattice Monte Carlo (not discussed here)

NOTATION: $S_0, \mathcal{L}_0, \langle\ \rangle_0, Z_0[J]$ refer free theory ($\lambda = 0$) quantities

<u>Perturbation theory</u>: we could write

$$Z[J] = \int \mathcal{D}\varphi\, e^{iS_0[J] + S_I} = \int \mathcal{D}\varphi\, e^{iS_0}\left(1 + iS_I + \frac{i^2}{2} S_I^2 + \dots\right)$$

in order to obtain pert. expansion for $Z[J]$ and for all Green functions. However, it is more convenient to do the following:

Note that

$$\left[\frac{\delta}{i\delta J(x)}\right]^n Z_0[J] = \int \mathcal{D}\varphi \left[\frac{\delta}{i\delta J(x)}\right]^n e^{iS_0[\varphi, J]}$$

$$= \int \mathcal{D}\varphi\, \varphi(x)^n e^{iS_0[\varphi, J]}$$

Thus, we can write

$$Z[J] = e^{iS_I\left[\frac{\delta}{i\delta J}\right]} Z_0[J] = e^{i\int d^4x\, \mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right)} Z_0[J] \qquad (2.34)$$

To order λ this is

$$Z[J] = Z_0[J] + i\int d^4x\, \mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right) Z_0[J] + \dots$$

$$= Z_0[J] - i\frac{\lambda}{4!}\int d^4x \left[\frac{\delta}{i\delta J(x)}\right]^4 Z_0[J] + \dots$$

Let us use notation $Z_0[J] = e^{-\frac{1}{2} J_x G_{xy} J_y}$

$$\frac{\delta}{\delta J_x} Z_0[J] = -G_{xy} J_y Z_0[J]$$

$$\left(\frac{\delta}{\delta J_x}\right)^2 Z_0[J] = \left(-G_{xx} + G_{xy} J_y G_{xz} J_z\right) Z_0[J]$$

$$\left(\frac{\delta}{\delta J_x}\right)^3 Z_0[J] = \left(+G_{xx} G_{xy} J_y + 2 G_{xx} G_{xy} J_y - G_{xy} J_y G_{xz} J_z G_{xr} J_r\right) Z_0[J]$$

$$\left(\frac{\delta}{\delta J_x}\right)^4 Z_0[J] = \left(3(G_{xx})^2 - 3G_{xx} G_{xy} J_y G_{xz} J_z - 3G_{xx} G_{xy} J_y G_{xz} J_z + G_{xy} J_y G_{xz} J_z G_{xr} J_r G_{xs} J_s\right) Z_0[J]$$

Thus, we obtain

$$\frac{Z[J]}{Z[0]} = \frac{1 - i\frac{\lambda}{4!}\int d^4x \left(3G_{xx}^2 - 6G_{xx}(G_{xy}J_y)^2 + (G_{xy}J_y)^4\right)}{1 - i\frac{\lambda}{4!}\int d^4x \, 3G_{xx}^2} \frac{Z_0[J]}{Z_0[0]}$$

$$= \left(1 - i\frac{\lambda}{4!}\int d^4x\left[-6G_{xx}(G_{xy}J_y)^2 + (G_{xy}J_y)^4\right]\right)\frac{Z_0[J]}{Z[J]} + \mathcal{O}(\lambda^2)$$

when we include the normalization $Z[0]$. (2.35)

Now, we can calculate n-point Green functions:

$$\langle\varphi(x)\rangle = \frac{\delta}{i\delta J_x}\frac{Z[J]}{Z[0]}\Bigg|_{J=0} = \underbrace{\left[\frac{\delta}{i\delta J_x}(\dots)\right]\Bigg|_{J=0}}_{=0}\frac{Z_0[J]}{Z_0[0]} + (\dots)\underbrace{\frac{\delta}{i\delta J_x}\frac{Z_0[J]}{Z_0[0]}\Bigg|_{J=0}}_{=0.} + \mathcal{O}(\lambda^2)$$

$$= 0.$$

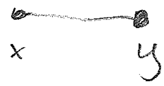
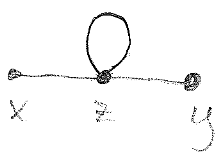
However, the 2-point function is non-trivial:

$$\frac{\delta^2}{i\delta J_x\, i\delta J_y}\frac{Z[J]}{Z[0]}\bigg|_{J=0} = \overset{1}{(\ldots)}\,\frac{\delta^2}{i\delta J_x\, i\delta J_y}\frac{Z_0[J]}{Z_0[0]}\bigg|_{J=0}$$

$$+\left[\frac{\delta^2}{i\delta J_x\, i\delta J_y}(\ldots)\right]\frac{Z_0[J]}{Z_0[0]}\bigg|_{J=0} + O(\lambda^2)$$

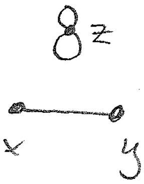
$$= G_F(x-y) - i\frac{\lambda}{4!}12\int d^4z\, G_{zz}G_{xz}G_{yz} + O(\lambda^2) \qquad (2.36)$$

Diagrammatically this is

$\langle \varphi(x)\varphi(y)\rangle =$  $- i\frac{\lambda}{2}$  $+ O(\lambda^2)$ (2.37)

These are examples of coordinate space (x-space) Feynman diagrams

NOTE: What happens if we do not normalize $\frac{Z[J]}{Z[0]}$ in (2.35)? Then we would have additional contribution $\int d^4x\, 3\, G_{xx}^2\, Z_0[J]$, giving additional Feynman diagram to (2.37):

$i\frac{\lambda}{4!}6$  $= i\frac{\lambda}{4!}6\, G_{xy}\int d^4z\, G_{zz}^2$