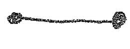
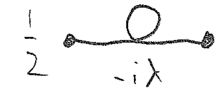
3.4. Renormalization of 2-point function

Consider again the Green function (2.37)

$$G^{(2)}(x,y) = \langle \varphi(x)\varphi(y)\rangle = $$  $+\frac{1}{2}$  $+ \mathcal{O}(\lambda^2)$

G_F $-i\lambda$

In momentum space this is (non-amputated)

$$\left(\tilde{G}_F(\bar{p}) = \frac{i}{k^2 - m^2 + i\epsilon} \rightarrow \frac{-i}{k^2+m^2} \;;\; d^4k \rightarrow i\, d^4k \right)$$

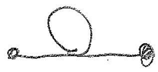
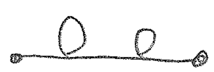
$$\langle \tilde{\varphi}(p)\tilde{\varphi}(q)\rangle = (2\pi)^4 \delta^{(4)}(p-q)\left(\tilde{G}_F(p) - i\tilde{G}_F(p)\,\Pi(p)\,\tilde{G}_F(p)\right)$$

where

$$\Pi(p) = \frac{\lambda}{2}\int \frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2+m^2} \quad (\text{p. } 83) \qquad (3.25)$$

(in this case, not a function of p)

For 2-pt. function, we can actually sum all contributions at 1-loop to full propagator:

$\langle \tilde{\varphi}(p)\tilde{\varphi}(q)\rangle =$  $+$  $+$  $+ \ldots$ (3.26)

$$= -i(2\pi)^4 \delta^{(4)}(p-q)\left(i\tilde{G} - (i\tilde{G})\Pi(i\tilde{G}) + (i\tilde{G})\Pi(i\tilde{G})\Pi(i\tilde{G}) + \ldots \right)$$

$$= -i(2\pi)^4 \delta^{(4)}(p-q)\,\frac{1}{(i\tilde{G}(p))^{-1} + \Pi(p)}$$

$$= -i(2\pi)^4 \delta^{(4)}(p-q)\,\frac{1}{p^2+m^2+\Pi(p)} \qquad (3.27)$$

Note: this is exactly like the free $\tilde{G}_F(p)$ (w. euclidean p), but with modification

$$m^2 \to m^2 + \Pi(p) \qquad (3.28)$$

$$= m^2 - \frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\mu^{-2\epsilon}\left(\frac{1}{\epsilon} + 1 - \gamma_E + \ln\frac{4\pi\mu^2}{m^2} + \mathcal{O}(\epsilon)\right) \qquad (3.29)$$

in $4-2\epsilon$ dimensions (3.24)

This diverges as $\epsilon \to 0$, as mentioned several times.

The point of renormalisation is that this divergence can be canceled by redefining $m^2 \to m^2 + \delta m^2$, where formally δm^2 is $\mathcal{O}(\lambda)$ and chosen to cancel the divergence in .

How to choose δm^2?

obviously, it has to be

$$\delta m^2 = m^2 \frac{\lambda}{2}\frac{1}{(4\pi)^2}\mu^{-2\epsilon}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)$$

Choice of δm^2: renormalization scheme

* In MS (Minimal Subtraction) scheme we subtract only $1/\epsilon$ - pole:

$$\delta m^2 = \frac{\lambda}{32\pi^2} m^2 \mu^{-2\epsilon} \frac{1}{\epsilon} \tag{3.30}$$

Now $m^2 + \Pi \to m^2 - \frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\mu^{-2\epsilon}\left(1 - \gamma_E + \ln\frac{4\pi\mu^2}{m^2} + O(\epsilon)\right) + O(\lambda^2)$

is finite.

* In modified MS-scheme, $\overline{MS}$, also γ_E and $\ln 4\pi$ are absorbed in δm^2:

$$\delta m^2 = \frac{\lambda}{32\pi^2} m^2 \mu^{-2\epsilon}\left(\frac{1}{\epsilon} - \gamma_E - \ln 4\pi\right) \tag{3.31}$$

now $m^2 + \Pi \to m^2 - \frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\mu^{-2\epsilon}\left(1 + \ln\frac{\mu^2}{m^2}\right) + O(\epsilon) + O(\lambda^2)$

This modification ($m^2 \to m^2 + \delta m^2$) makes the 1-loop correction to become finite; infinity is $O(\lambda^2)$

Theory where this can be done order-by-order is renormalizable.

3.5 Interpretation :

- Let us denote the original mass^2 which appears in the Lagrangian bare mass^2, m_B^2.

- We measure (or calculate) the 2-point function $\langle\tilde{\varphi}(p)\tilde{\varphi}(q)\rangle$. This is a physical and finite quantity, which behaves at small p as

$$\propto \delta^{(4)}(p-q)\,\frac{1}{p^2+m_{\text{Pole}}^2} \qquad (3.32)$$

where m_{Pole} is the "pole mass", physical pole of the propagator, i.e. the observed mass of the particle.

- However, when we do the loop calculation using bare mass, we have

$$\langle\tilde{\varphi}(p)\tilde{\varphi}(q)\rangle \propto \frac{1}{p^2+m_B^2+\Pi} \qquad (3.33)$$

$$\text{"}-\frac{\lambda_B}{2}\frac{m_B^2}{(4\pi)^2}\tilde{\mu}^{2\epsilon}\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\text{"}$$

The loop correction Π diverges; thus, in order for the physical 2-pt. function to remain finite, m_B^2 must diverge too!

Thus, bare (Lagrangian) mass is not physical

- We obtain renormalized mass², m_R^2, by subtracting the divergence:

$$\underline{m_B^2 = m_R^2 + \delta m^2} \equiv \underline{Z_{m^2} m_R^2} \tag{3.34}$$

m_R^2 is finite, but scheme-dependent. (MS, $\overline{\text{MS}}$)

m^2 appearing in (3.29)–(3.31) is actually m_R^2, because it is finite.

This actually generalizes to λ and the field φ itself: Lagrangian is defined in terms of bare quantities,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_B)(\partial^\mu \varphi_B) - \frac{1}{2} m_B^2 \varphi_B^2 - \frac{1}{4!}\lambda_B \varphi_B^4 \tag{3.35}$$

These are "naked", "bare" quantities, which are not observable.

We can define renormalized quantities through

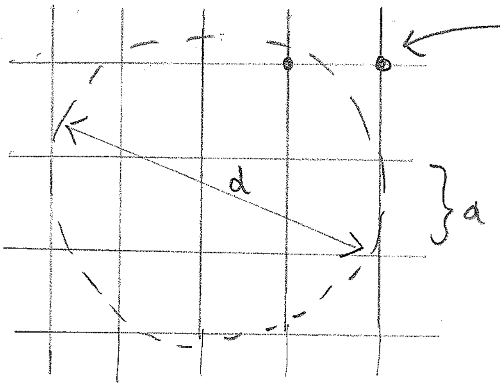
$$\begin{cases} \varphi_B = Z_\varphi^{1/2} \varphi_R \\ m_B^2 = Z_{m^2} m_R^2 = m_R^2 + \delta m^2 \\ \lambda_B = Z_\lambda \lambda_R = \lambda_R + \delta\lambda \end{cases} \tag{3.36}$$

Because free theory has no divergences,

$$Z_i = 1 + \mathcal{O}(\lambda_R)$$

Dimensional regularization makes the renormalization somewhat muddy.

A particularly transparent view is offered by lattice regularization: define theory on discrete lattice

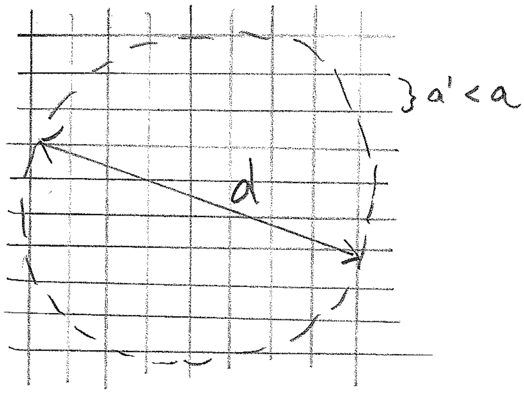


φ_B, m_B^2, λ_B lives on lattice scale

φ_R: average of φ_B over distance scale $d \sim \frac{1}{\mu}$

m_R^2, λ_R : Measure $G^{(2)}$, $G^{(4)}$ when $|x-y| \sim d$.

go towards continuum



- Renormalized quantities kept constant

$\Rightarrow$

Bare φ_B, m_B^2, λ_B change!

- m_B^2, λ_B diverge as $a \to 0$, but long-distance physics (λ_R, m_R, φ_R) constant!

$m_B^2 - m_R^2 - m_{Pole}^2$?

From page 88-89, and (3.21)

$$m_{Pole}^2 = m_B^2 - \frac{\lambda_B}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2+m_B^2} \qquad m_B \sim \frac{1}{\epsilon}$$

$$= m_R^2 + \delta m^2 - \frac{\lambda_R}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2+m_R^2} + \mathcal{O}(\lambda_R^2)$$

Use MS,

$$\delta m^2 = \frac{\lambda \mu^{-2\epsilon}}{32\pi^2} m_R^2 \frac{1}{\epsilon}$$

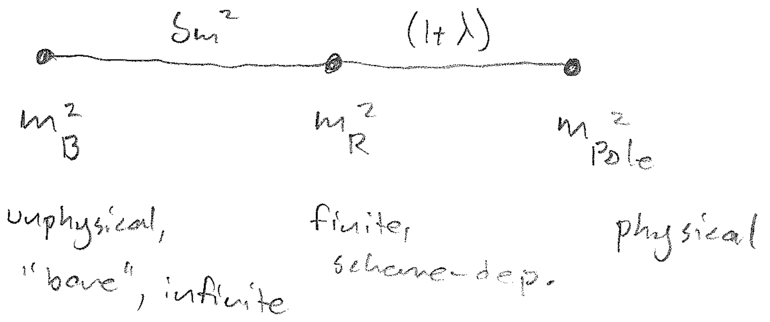
switching $\begin{cases} \lambda_B \to \lambda_R \\ m_B^2 \to m_R^2 \end{cases}$ causes a difference $\mathcal{O}(\lambda^2)$!

$$= m_R^2\left(1 - \frac{\lambda_R \mu^{-2\epsilon}}{32\pi^2}\left(1 - \gamma_E + \ln\frac{4\pi\mu^2}{m_R^2} + \mathcal{O}(\epsilon)\right) + \mathcal{O}(\lambda_R^2)\right) \tag{3.37}$$

Thus, m_R^2 depends on the schema (MS, $\overline{\text{MS}}$, lattice, etc...) but also on the choice of the <u>renormalization scale</u> μ (in order for m_{Pole}^2 to be constant).

Often $\underline{\lambda_R \mu^{-2\epsilon}}$ is denoted by $\underline{\lambda_R}$

(natural dimensions in $d=4-2\epsilon$ dimensions)



3.6 Renormalization of Green functions

The definition follows from B ⇔ R relations on page 89:

$$G^{(n)}_{B,c}(x_1 .. x_n) = \langle \varphi_B(x_1) .. \varphi_B(x_n) \rangle_c$$

$$G^{(n)}_{R,c}(x_1 .. x_n) = \langle \varphi_R(x_1) .. \varphi_R(x_n) \rangle_c$$

$$\Rightarrow \quad \underline{G^{(n)}_B(x_1 .. x_n) = Z_\varphi^{n/2} \, G^{(n)}_R(x_1 .. x_n)} \qquad (3.38)$$

This generalizes to Fourier-transformed Green functions

$$\underline{\tilde{G}^{(n)}_{B,c} = Z_\varphi^{n/2} \, \tilde{G}^{(n)}_{R,c}} \qquad (3.39)$$

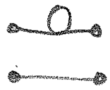
For amputated Green functions we obtain

$$\underline{\tilde{G}^{(n)}_{B,\bar{c}}} = [\tilde{G}^{(2)}_B(p_1)]^{-1} \cdots [\tilde{G}^{(2)}_B(p_n)]^{-1} \, \tilde{G}^{(n)}_{B,c}$$

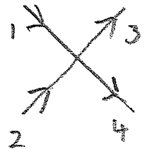
$$\underline{= Z_\varphi^{-n/2} \, \tilde{G}^{(n)}_{R,\bar{c}}} \qquad (3.40)$$

3.7. Renormalized λ_R to 1-loop

λ_R can be obtained from connected & amputated Green 4-pt. function when all external $p_i = 0$.

[why connected? terms like  contribute to mass^2 renormalization, not λ!]

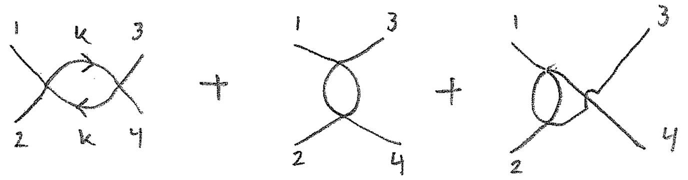
- Lowest order connected:

$$\tilde{G}^{(4)}_{B,\tilde{c}}(p_1,p_2,p_3,p_4)\Big|_{\lambda} =$$  $$= -i\lambda_B \times (2\pi)^4\,\delta(p_1+p_2-p_3-p_4)$$

(Feynman rules, 2.48)

$$\Rightarrow \tilde{G}^{(4)}_{B,\tilde{c}}(0,0,0,0) = -i\lambda_B \times (2\pi)^4\,\delta^{(4)}(0) \qquad (3.41)$$

- 2nd order:

$$\tilde{G}^{(4)}_{B,\tilde{c}}\Big|_{\lambda^2} =$$ 

3 non-equivalent permutations of external legs

$$p_i = 0 \rightarrow \quad = 3\cdot S\cdot(-i\lambda_B)^2\int\frac{i\,d^4k}{(2\pi)^4}\left[\frac{-i}{k^2+m_B^2}\right]^2 \qquad \text{euclidean!}$$

S: $\frac{1}{2}$, page 67

$$\times(2\pi)^4\,\delta^{(4)}(0)$$

$$= i\,\frac{3}{2}\,\lambda_B^2\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+m^2)^2}\times(2\pi)^4\,\delta^{(4)}(0) \qquad (3.42)$$

Go to $d = 4 - 2\epsilon$ dimensions.

Now

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+m^2)^2} = -\frac{d}{dm^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2}$$

$$= -\frac{d}{dm^2}\left(-\frac{m^2}{(4\pi)^2}\mu^{-2\epsilon}\left(\frac{1}{\epsilon} + 1 - \gamma_E + \ln\frac{4\pi\mu^2}{m^2} + O(\epsilon)\right)\right)$$

(3.24)

$$= \frac{\mu^{-2\epsilon}}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m^2} + O(\epsilon)\right) \tag{3.43}$$

Thus,

$$\tilde{G}^{(4)}_{B,\tau}(0,0,0,0) = -i\left[\lambda_B - \frac{3}{2}\lambda_B^2 \frac{\mu^{-2\epsilon}}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m_B^2} + O(\epsilon)\right)\right]$$
$$\times (2\pi)^d \delta^{(4)}(0) + O(\lambda_B^3) \tag{3.44}$$

Now, using (3.40) and $Z_\varphi = 1 + O(\lambda_R^2)$ (Homework!)

we have $\tilde{G}^{(4)}_{B,\tau} = \tilde{G}^{(4)}_{R,\tau} + O(\lambda_R^3)$.

Using $\lambda_B = Z_\lambda \lambda_R = \lambda_R + \underbrace{\delta\lambda}_{O(\lambda^2)}$

$$\tilde{G}^{(4)}_{R,\tau}(0,0,0,0) = -i\left[\lambda_R + \delta\lambda - \frac{3}{2}\frac{\lambda_R^2\mu^{-2\epsilon}}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m_R^2} + O(\epsilon)\right)\right]$$
$$\times (2\pi)^d \delta^{(d)}(0) + O(\lambda_R^3) \tag{3.45}$$

Again, switching $\lambda_B \to \lambda_R$; $m_B \to m_R$ in the last piece gives higher order $O(\lambda^3)$ contribution!

Like for 2-point function, we say

$$\tilde{G}^{(4)}_{R,\bar{c}} = -i\lambda_{\text{Phys}} \times (2\pi)^4 \delta^{(4)}(0) \qquad (3.46)$$

For this to be finite as $\epsilon \to 0$, we must have

$$\delta\lambda = \frac{3}{2}\frac{\lambda_R^2 \mu^{-2\epsilon}}{(4\pi)^2}\frac{1}{\epsilon} \qquad (3.47)$$

("minimal subtraction"), and λ_B diverges as $\frac{1}{\epsilon}$

3.8. Renormalization group

- Renormalization group tells us how the parameters have to be changed (when we change the scale) in order for physics to remain the same
- In our lattice RG example on page 90: how to change λ_B, m_B^2, etc. in order to keep physics constant at long distances when we change scale?
- Usually expressed as differential Callan-Symanzik-equations

$$m_R^2 \frac{d}{dm_R^2} G^{(n)}_{R,\epsilon} = \ldots$$

involving only renormalized quantities.

In dimensional regularization, the dependence on the regularization scale μ is unphysical and must vanish for physical quantities:

$$\mu \frac{d}{d\mu} \tilde{G}^{(n)}_{R,\bar{c}} = 0 = \left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial m_R^2}{\partial \mu} \frac{\partial}{\partial m_R^2} + \mu \frac{\partial \lambda_R}{\partial \mu} \frac{\partial}{\partial \lambda_R} \right) \tilde{G}^{(n)}_{R,\bar{c}}$$

(where "0" can mean of higher order in λ)

Applying to our result (3.45), (3.46)

$$\lambda_{phys} = \lambda_R - \frac{3}{2} \frac{\lambda_R^2 \mu^{-2\epsilon}}{(4\pi)^2} \left(-\gamma_E + \ln \frac{4\pi \mu^2}{m_R^2} \right) + O(\lambda_R^3) \quad \Big| \cdot \mu \frac{\partial}{\partial \mu}$$

$$\Rightarrow \quad 0 = \mu \frac{d}{d\mu} \lambda_R - 3 \frac{\lambda_R^2 \mu^{-2\epsilon}}{(4\pi)^2} + O(\lambda_R^3, \epsilon \lambda_R^2)$$

$$\Rightarrow \quad \underline{\mu \frac{d}{d\mu} \lambda_R = \frac{3}{(4\pi)^2} \lambda_R^2} \qquad (3.48)$$

We could apply this to λ_B too:

$$\lambda_B = \lambda_R + \delta\lambda = \lambda_R + \frac{3}{2} \frac{\lambda_R^2 \mu^{-2\epsilon}}{(4\pi)^2} \frac{1}{\epsilon} \quad \Big| \cdot \mu \frac{d}{d\mu} \qquad (3.49)$$

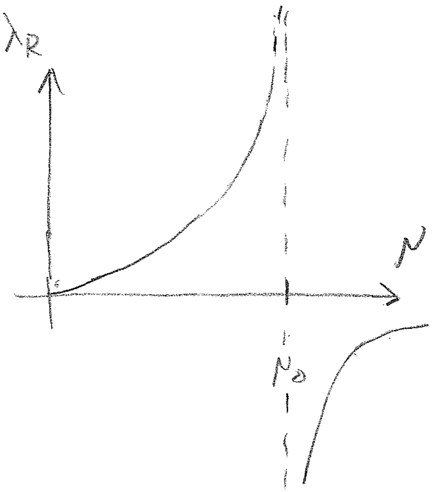
$$\Rightarrow 0 = \mu \frac{d}{d\mu} \lambda_R - 3 \frac{\lambda_R^2 \mu^{-2\epsilon}}{(4\pi)^2}$$

$$\Rightarrow \underline{\mu \frac{d}{d\mu} \lambda_R = \frac{3}{(4\pi)^2} \lambda_R^2} \quad \text{as before!}$$

It is often easier to obtain RG-equations than the full quantity to any given order.
It is sufficient to obtain the $\frac{1}{\epsilon}$-piece (2nd example)!

Solution of (3.48):

$$\lambda_R(\mu) = \frac{(4\pi)^2}{3} \frac{1}{\ln \frac{\mu_0}{\mu}} \qquad (3.50)$$



* Solution is reliable only if $\lambda_R \ll 1$, i.e. $\mu \ll \mu_0$.

* As μ increases, λ_R grows: interactions become stronger

* We should choose μ to minimize corrections between λ_{phys} and λ_R: (3.45) $\Rightarrow$ $\mu^2 \sim m_R^2$, and $\lambda_{phys} \approx \lambda_R$.

* If we were to calculate $\tilde{G}^{(4)}$ with finite p, we would obtain $\mu \sim |p|$ in order for $\lambda_R \simeq \lambda_{phys}$. This means that λ_{phys} increases with increasing $|p|$!

* If we require the theory to be valid at <u>all</u> p, we need to take $\mu_0 \to \infty$ ($\mu_0 > |p|$). This indicates that λ_R and $\lambda_{phys} \to 0$ for any fixed p.

$\Rightarrow$ Theory is <u>trivial</u>, non-interacting.

* Another option: theory is valid only for $|p|$ up to some $|p|_{max} < \mu_0$.

After this, some <u>new theory</u> takes over! Our $\lambda\varphi^4$ is only an effective low-energy theory of the true high-energy one.

This is the case with the Standard Model!

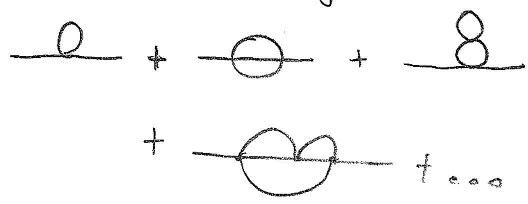
"μ_0" depends on the Higgs mass.

Note: on p. 87 we renormalized 2-point function (m^2) to 1-loop order. The result to <u>all</u> (perturbative) orders is

$\langle \tilde{\varphi}(p)\tilde{\varphi}(q)\rangle =$ 

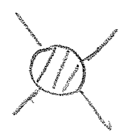
$$= -i(2\pi)^4\,\delta^{(4)}(p-q)\,\frac{1}{p^2+m^2+\Pi(p)}$$

where now $\Pi(p) =$  $=$ sum of all 1PI diagrams, e.g.

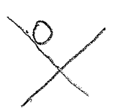
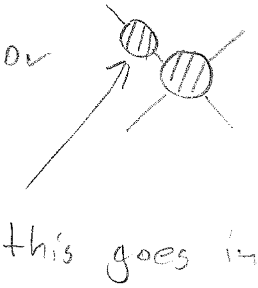


Lots of divergences, which can all be pushed to higher orders!

Likewise, the full expression for the 4-pt function is

$\tilde{G}^{(4)}_c =$  $= \sum$ 1PI 4-pt. diagrams

but not 1P-reducible diagrams as

 , or 

this goes into renormalization of the ext. line!