4. Fermions

4.1. Free fermions: described by

a 4-component complex vector ψ, which obeys the Dirac equation

$$(i\not\partial - m)\psi = 0 \qquad (4.1)$$

where $\not\partial \equiv \gamma^\mu \partial_\mu$ (in general, $\not a = \gamma^\mu a_\mu$)

and γ_μ are 4×4-matrices which obey 4-dimensional Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\cdot\mathbb{1} \qquad (4.2)$$

- γ's are not unique; we shall use here Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad (4.3)$$

where $\mathbb{1}$ is 2×2 unit matrix and σ^i are Pauli σ-matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (4.4)$$

These obey $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$ (4.5)

$$\Rightarrow \sigma^k = -\frac{i}{2}\epsilon^{kij}\sigma^i\sigma^j \qquad (4.6)$$

Note that if $\psi(x)$ is a solution of the Dirac equ, it is also a solution of Klein-Gordon equation: multiply Dirac equ. by $(-i\not\partial - m)$:

$$0 = (-i\not\partial - m)(i\not\partial - m)\psi$$
$$= \left(\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu - m^2\right)\psi = (\partial_\mu\partial^\mu - m^2)\psi \qquad (4.7)$$

Effectively "Dirac $\hat{=} \sqrt{\text{K-G}}$"

Historically, Dirac derived (4.1) because K-G equ. is 2nd order in time, and $\rho = |\varphi|^2$ is not conserved (no cons. probability ?) $\Rightarrow$ how to get 1st order ? (see particle physics notes, p. 83)

ψ transforms as spin-½ representation in Lorentz-transformations:

Spin-1 (vector): $V^\mu \to \Lambda^\mu{}_\nu V^\nu \qquad (4.8)$

spin-½ (spinor): $\psi \to \Lambda_{1/2}\psi, \qquad (4.9)$

where $\Lambda_{1/2} = \exp\left(-\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$

$S^{\mu\nu} = \tfrac{i}{4}[\gamma^\mu,\gamma^\nu]$

parameters of Lorentz-transformation

$$\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu \qquad (4.10)$$

$\Rightarrow (i\not\partial - m)\psi = 0$ Lorentz-invariant

Lagrangian :

Introduce $\boxed{\bar{\psi} \equiv \psi^\dagger \gamma^0}$, and let us consider $\psi^\dagger$ and ψ independent (as z, z^* in complex analysis. Now if

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \tag{4.11}$$

we obtain Dirac eqn. through

$$\frac{\delta \mathcal{L}}{\delta \bar{\psi}} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\psi})} = 0 \tag{4.12}$$

• $\mathcal{L}$ is Lorentz invariant (check!) Is $\psi^\dagger \psi$?

Free fermions :

Consider plane waves: let

$$\psi = u(p,s)\, e^{-ip\cdot x} \quad , \quad p^2 = m^2 , \; p^0 > 0 \tag{4.13}$$

where $u(p,s)$ has 4 components (s = spin)

$\Rightarrow$

$$(\not{p} - m)\, u(p,s) = 0 \tag{4.14}$$

Correspondingly, if $\psi = v(p,s)\, e^{ip\cdot x}$, $p^2 = m^2$, $p^0 > 0$

$\Rightarrow$

$$(\not{p} + m)\, v(p,s) = 0 \tag{4.15}$$

Here u, v will describe particles and antiparticles

- Denoting $\underline{\sigma = (\mathbb{1}, \vec{\sigma})}$; $\underline{\bar{\sigma} = (\mathbb{1}, -\vec{\sigma})}$ (4.16)

(4.14) becomes

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u(p,s) = 0 \quad (4.17)$$

Because $(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0)^2 - p^i p^j \sigma^i \sigma^j = p^2 = m^2$,
the solution of (4.17) is

$$u(p,s) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi_s \\ \sqrt{p\cdot\bar{\sigma}}\ \xi_s \end{pmatrix} \quad (4.18)$$

where ξ_s is a 2-component vector. Similarly,
(4.15) $\Rightarrow$

$$\begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} v(p,s) = 0 \Rightarrow v(p,s) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\ \eta_s \end{pmatrix} \quad (4.19)$$

where η_s is 2-comp. vector.

- Here ξ_s (and η_s) is <u>spin vector</u>: the operator $\underline{\frac{1}{2}\hat{s}\cdot\vec{\sigma}}$ $\quad |\hat{s}|=1$, $\hat{s}$ meas. direction

has 2 eigenvalues $\pm\frac{1}{2}$. For example,
choosing $\hat{s} = \hat{z} \Rightarrow \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,
eigenvectors $\xi_{+1} \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\xi_{-1} \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Normalization

- Normalization of spinors varies; we use here

$$\begin{cases} \bar{u}(p,s)\, u(p,s') = 2m\, \delta_{ss'} & (4.20) \\ \bar{v}(p,s)\, v(p,s') = -2m\, \delta_{ss'} & (4.21) \end{cases}$$

This implies that $\xi^{+}\xi = \eta^{+}\eta = 1$ (4.22)

- The vectors are orthogonal:

$$u^{+}(p,s)\, v(-p,s') = v^{+}(p,s)\, u(-p,s') = 0 \qquad (4.23)$$

- From (4.20), (4.21) $\Rightarrow$

$$\begin{cases} u^{+}(p,s)\, u(p,s') = 2E_{\bar{p}}\, \delta_{ss'} \\ v^{+}(p,s)\, v(p,s') = 2E_{\bar{p}}\, \delta_{ss'} \end{cases} \qquad E_{\bar{p}} = p^{0} = \sqrt{\bar{p}^{2}+m^{2}} \qquad (4.24)$$

- Completeness relation; spin summation:

- $$\sum_{s=\pm 1} u(\bar{p},s)\, \bar{u}(\bar{p},s) = 4\times 4\text{-matrix}$$

$$= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = \not{p} + m \qquad (4.25)$$

because $\sum_{s} \xi_s \xi_s^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Likewise,

- $$\sum_{s=\pm 1} v(p,s)\, \bar{v}(p,s) = \not{p} - m \qquad (4.26)$$

The completeness relations are often used in computations.

4.2. Quantization of fermion fields

- Fermions can be quantized through 2nd quantization, as we did for Klein-Gordon field: (see particle physics notes, p. 99)

$$\hat{\psi} = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \sum_{s=\pm1} \left[\hat{a}^{(s)}_{\bar{p}} \, u(\bar{p},s) e^{-ip\cdot x} + \hat{b}^{+(s)}_{\bar{p}} \, v(\bar{p},s) \, e^{ip\cdot x} \right]$$

$$\hat{\bar{\psi}} = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \sum_{s=\pm1} \left[\hat{a}^{+(s)}_{\bar{p}} \, \bar{u}(\bar{p},s) e^{ip\cdot x} + \hat{b}^{(s)}_{\bar{p}} \, \bar{v}(\bar{p},s) \, e^{-ip\cdot x} \right] \qquad (4.27)$$

here $\hat{a}, \hat{b}$ anticommute:

$$\left\{ \hat{a}^{(s)}_{\bar{p}}, \hat{a}^{+(s')}_{\bar{p}'} \right\} = \left\{ \hat{b}^{(s)}_{\bar{p}}, \hat{b}^{+(s')}_{\bar{p}'} \right\} = \delta^{(3)}(\bar{p}-\bar{p}') \, \delta_{ss'}$$

$$\{ \hat{a}, \hat{a} \} = \{ \hat{b}, \hat{b} \} = 0 \qquad (4.28)$$

Hamilton operator

$$\hat{H} = \int d^3\bar{p} \, E_{\bar{p}} \sum_{s=\pm1} \Big[\underbrace{\hat{a}^{+(s)}_{\bar{p}} \, \hat{a}^{(s)}_{\bar{p}}}_{\hat{n}^{(s)}_{\bar{p}}} + \hat{b}^{+(s)}_{\bar{p}} \, \hat{b}^{(s)}_{\bar{p}} - \delta^{(3)}(0) \Big]$$

0-point energy, $-\infty$!

$\hat{a}^+$: creates particle

$\hat{a}$: annihilates particle

$\hat{b}^+$: creates antiparticle

$\hat{b}$: annihilates antiparticle

What we shall use here is path integral formalism for fermions

For gauge field theories (= all physical theories) path integral is the preferred method

4.3. Path integral for fermions

- Recall: for bosons, path integral over classical fields → quantum field theory
- What are "classical" fermion fields?
- It works with anticommuting Grassmann variables (Berezin 1966)
- let $\{c_i\}$ be a set of Grassmann-numbers:

$$c_i c_j = -c_j c_i \Leftrightarrow \{c_i, c_j\} = 0 \qquad (4.29)$$

$\Rightarrow$ $c^2 = 0$ for Grassmann-numbers

- For any function f:

$$f(c) = f(0) + f'(0)\, c = A + B \cdot c \qquad (4.30)$$

- If f is a function of N variables

$$f(c_1, \dots, c_N) = f^{(0)} + f^{(1)}_i c_i + f^{(2)}_{ij} c_i c_j + \dots + f^{(N)} c_1 c_2 \dots c_N \qquad (4.31)$$

- Define derivative operator:

$$\left\{ \frac{\partial}{\partial c_i}, c_j \right\} = \delta_{ij} \quad ; \quad \left\{ \frac{\partial}{\partial c_i}, \frac{\partial}{\partial c_j} \right\} = 0 \qquad (4.32)$$

- And integral

$$\int dc_i = 0 \quad , \quad \int dc_i\, c_j = \delta_{ij} \qquad (4.33)$$

Note order of integration:

$$\int dc_1\, dc_2\, c_2 c_1 = 1 = -\int dc_2\, dc_1\, c_2 c_1 \qquad (4.34)$$

We can define now <u>complex</u> Grassman numbers:

θ, θ^* are <u>independent</u> Grassmann-numbers:

$$\begin{aligned} \theta^*\theta &= -\theta\theta^* \\ (\theta\eta)^* &\equiv \eta^*\theta^* = -\theta^*\eta^* \\ (\theta^*)^* &= \theta \\ \theta^2 &= \theta^{*2} = 0 \end{aligned} \qquad (4.35)$$

Thus, the "gaussian" integral is

$$\underline{\int d\theta^*\, d\theta\; e^{-b\theta^*\theta}} = \int d\theta^* d\theta \left(1 - b\theta^*\theta\right) = \underline{b} \qquad (4.36)$$

$$= -\int d\theta\, d\theta^*\, e^{-b\theta^*\theta}$$

Compare with $\int dz^* dz\, e^{-bz^*z} = \frac{\pi}{b}$, $z \in \mathbb{C}$

If M is $N\times N$-matrix, we have

$$\underline{\int \Big[\prod_i^N d\theta_i^* d\theta_i\Big] e^{-\theta_i^* M_{ij}\theta_j}}$$

$$= \int \Big[\prod d\theta_i^* d\theta_i\Big] \left(\overset{0}{\cancel{1}} \overset{0}{\cancel{- \theta_i^* M\theta}} + \ldots + \frac{(-1)^N}{N!}\left(\theta_i^* M_{ij}\theta_j\right)^N\right)$$

$$= \int \Big[\prod d\theta_i^* d\theta_i\Big] (-1)^N \left(\theta_i^* M_{i1}\theta_1\right)\left(\theta_j^* M_{j2}\theta_2\right)\ldots\left(\theta_l^* M_{lN}\theta_N\right)$$

because $(\theta_a^*\theta_b)$, $(\theta_c^*\theta_d)$ commute

$$= \int \Big[\prod_i d\theta_i^*\Big] (-1)^{[\frac{N}{2}]}\, \theta_i^* M_{i1}\, \theta_j^* M_{j2} \ldots \theta_l^* M_{lN}$$

where $[N/2]$ = integer part of $N/2$

$$= \int [\prod_i d\theta_i^*]\, \theta_1^* \theta_2^* \cdots \theta_N^* \; \epsilon^{abc\ldots e} M_{a1} M_{b2} \cdots M_{eN} \times (-1)^{[N/2]}$$

$$= \epsilon^{abc\ldots} M_{a1} M_{b2} M_{c3} \cdots = \underline{\underline{\det M}} \qquad (4.37)$$

(Note: here we have $\prod_{l=0}^{N-1} (-1)^l = (-1)^{[N/2]}$ twice)

Thus, we obtain the important result

$$\underbrace{\int [\prod_{i=1}^{N} d\theta_i^* d\theta_i]\, e^{-\theta^{*T} M \theta} = \det M} \qquad (4.38)$$

We can also calculate

$$Z[\chi, \chi^*] = \int [\prod d\theta^* d\theta]\, e^{-\theta^{*T} M \theta - \theta^{*T} \chi - \chi^{*T} \theta} \qquad (4.39)$$

χ, χ^* Grassmann

Note that $\int d\theta\, f(\theta + \chi) = \int d\theta\, f(\theta)$, thus,

we can change variables as usual $\theta' = \theta + \chi$ etc.

Thus,

$$Z[\chi, \chi^*] = \int [\prod d\theta^* d\theta]\, e^{-(\theta^{*T} + \chi^{*T} M^{-1}) M (\theta + M^{-1}\chi) + \chi^{*T} M^{-1} \chi}$$

$$= \int [\prod d\theta^* d\theta]\, e^{-\theta^{*T} M \theta + \chi^{*T} M^{-1} \chi}$$

$$= \det M \times e^{\chi^{\dagger} M^{-1} \chi}, \qquad \chi^{\dagger} \equiv \chi^{*T}$$