1. **Momentary electric field**
   Show that the Meissner effect causes momentarily $\boldsymbol{E} \neq 0$ in superconductors.

2. **Low temperature specific heat**
   Consider a two-state system, in which the Hamiltonian has only two eigenstates with an energy difference Δ. By first calculating the free energy F, show that the specific heat of the system is

$$C = \frac{\Delta^2 e^{\frac{\Delta}{k_B T}}}{k_B T^2 \left(1 + e^{\frac{\Delta}{k_B T}}\right)^2}.$$

   Show that at low temperatures this behaves as

$$C \propto e^{-\frac{\Delta}{k_B T}}.$$

   (Hint: you can use Mathematica for derivations.)

3. **Specific heat in normal and superconducting states**
   (a) At low temperatures, the specific heat of a normal state metal is linear in temperature $C_\mathrm{n}(T) = \gamma T$. Using the third law of thermodynamics (i. e. the entropy has to vanish at $T = 0$) calculate the entropy of the normal state.
   (b) A phase transition is said to be of order n if the n:th derivative of $F(T)$ is discontinuous at T_C but lower derivatives as well of $F(T)$ are continuous.
      (i) Show that there is a latent heat associated with a first order transition (such as melting), but a second order transition has discontinuity of specific heat $C(T)$ but no latent heat.
      (ii) Thus deduce that the superconducting transition is of second order.
   (c) Show that the specific heat in the superconducting state $C_\mathrm{s}(T)$ has to satisfy

$$\int_0^{T_\mathrm{c}} \frac{C_\mathrm{s}(T)}{T}\, \mathrm{d}T = \gamma T_\mathrm{c}.$$

4. **Basic relations for Helmholtz free energy**
   In the lecture notes it is explained how the thermodynamic relations

$$F = E - ST, \qquad dF = -SdT - PdV, \qquad dE = TdS - PdV$$

   follow from the Gibbs distribution for a system in equilibrium with a heat bath at temperature T. Show this in detail.

1. **Basic formulas for the grand potential**
   Using the approach given in lecture notes in connection of F, derive the equations

$$\Omega = -\frac{1}{\beta}\ln[\mathrm{Tr}(e^{-\beta(\hat{H}-\mu\hat{N})})], \qquad \Omega = E - \mu N - ST, \qquad d\Omega = -SdT - pdV - Nd\mu.$$

2. **Ideal Bose gas**
   Calculate Ω for an ideal Bose-gas similarly as it is calculated for an ideal Fermi-gas in the lecture notes. Show that Bose and Fermi distributions reduce to a classical Maxwell-Boltzmann distribution

$$f(\epsilon) = e^{-\beta(\epsilon-\mu)}, \qquad \text{when} \qquad \beta(\epsilon-\mu) \gg 1.$$

   (Hint: The only difference in the calculation is that for bosons $n_\alpha = 0, 1, 2, \ldots$ instead of $n_\alpha = 0, 1$.)

3. **Ideal Fermi gas starting from Ω**
   Using the grand potential of an ideal Fermi-gas

$$\Omega = -k_B T \sum_\alpha \ln[1 + e^{\beta(\mu-\epsilon_\alpha)}],$$

   (a) verify that

$$N = -\frac{\partial\Omega}{\partial\mu} = \sum_\alpha f_\alpha \qquad \text{where} \qquad f_\alpha = \frac{1}{e^{\beta(\epsilon_\alpha-\mu)}+1}.$$

   (b) Calculate $S = -\partial\Omega/\partial T$ and show that it can be written in the form

$$S = -k_B \sum_\alpha [f_\alpha \ln f_\alpha + (1-f_\alpha)\ln(1-f_\alpha)]. \tag{1}$$

   [Hint: As an intermediate result derive

$$S = -k_B \sum_\alpha [\ln(1-f_\alpha) + \beta(\mu-\epsilon_\alpha)f_\alpha].$$

   and then seek how to write this into the form (1). Note: Equation (1) can also be derived directly from the definition of entropy, and thus is valid also in non-equilibrium cases.]

   (c) Using $E = \Omega + ST + \mu N$, show that

$$E = \sum_\alpha \epsilon_\alpha f_\alpha.$$

(d) Using $p = -\partial\Omega/\partial V$, show that

$$p = -\sum_\alpha \frac{\partial \epsilon_\alpha}{\partial V} f_\alpha.$$

(e) Starting from (1) show that

$$dS = \frac{1}{T}\sum_\alpha (\epsilon_\alpha - \mu) df_\alpha.$$

(f) Using the previous results show that

$$dE = TdS - pdV + \mu dN = \sum_\alpha (\epsilon_\alpha df_\alpha + f_\alpha d\epsilon_\alpha).$$

Notice that the change of the occupations (the first term) comes from changes in S and N (TdS and μdN terms), whereas the change of level energies comes from the change in V ($-pdV$ term).

4. **Ideal Fermi gas: pressure, entropy and energy**
Show that an ideal (spin 1/2) Fermi-gas has a pressure

$$p = \frac{2}{3}\frac{E}{V},$$

at all temperatures and at $T = 0$

$$S = 0 \qquad \text{and} \qquad \frac{E}{N} = \frac{3}{5}\epsilon_F.$$

5. **Level occupation directly from operator** (Optional exercise, no points)
In the lecture notes we identified the Fermi distribution from the expression for the total particle number for noninteracting fermions. Show, using a similar calculation, that the Fermi and Bose distributions follow also directly from

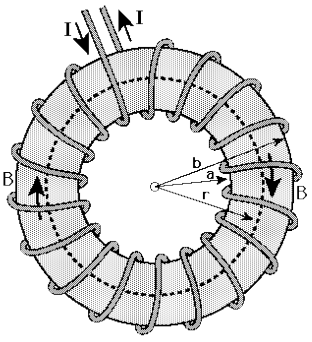
$$\langle \hat{n}_\alpha \rangle = \mathrm{Tr}[\hat{n}_\alpha e^{\beta(\Omega - \hat{H} + \mu \hat{N})}] = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} \pm 1}$$

Here $\hat{H} = \sum_\alpha \epsilon_\alpha \hat{n}_\alpha$ and $\hat{N} = \sum_\alpha \hat{n}_\alpha$, where $\hat{n}_\alpha$ is the number operator for the single-particle level α and ϵ_α is its energy. For fermions (upper sign) $\hat{n}_\alpha$ has the eigenvalues $n_\alpha = 0, 1$ and for bosons (lower sign) $n_\alpha = 0, 1, 2, \ldots$.

1. **Energy conservation in electromagnetic field**
   Derive the equation

$$-\,\mathrm{d}t \oint \mathrm{d}\boldsymbol{a}\cdot(\boldsymbol{E}\times\boldsymbol{H}) = \int dV(\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{D} + \boldsymbol{H}\cdot\mathrm{d}\boldsymbol{B} + \boldsymbol{E}\cdot\boldsymbol{j}_F\,\mathrm{d}t\,).$$

2. **Work done by a current source**
   In the lectures it was claimed that when the magnetic flux density $\boldsymbol{B}$ inside the sample changes, the current source has to do a work $V\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{B}$ in order to keep the current in the coil constant. Prove this more accurately. In order for the field to be homogeneous inside the sample (with $b-a \ll r$) and zero elsewhere, it is easiest to think of a toroidal coil surrounding a toroidal sample (see figure).



3. **Normal-superconducting transition**
   Calculate the latent heat and the change in the specific heat in a normal metal-superconductor phase transition, using

$$H_c(T) = H_c(0)\left[1-\left(\frac{T}{T_c}\right)^2\right].$$

   Sketch the results as a function of temperature.

4. **Critical current in a superconducting wire**
   Consider a superconducting wire with a radius R. Calculate the maximum supercurrent I_c that can flow so that the field caused by the current itself does not exceed the critical field H_c at the surface of the wire.

5. **Fermi temperature**
   Estimate the Fermi temperature for aluminum using the mass density $\rho = 2.7$ g/cm^3, the atomic weight and assuming 3 (noninteracting) conducting electrons/atom with an effective mass of a free electron m_e. Calculate the ratio T_c/T_F.

1. **Momentum integration**
   Show that for an arbitrary function $g(\boldsymbol{k})$
$$\frac{1}{L^3}\sum_{\boldsymbol{k}} g(\boldsymbol{k}) = \int \frac{d^2\Omega}{4\pi}\int d\epsilon N(\epsilon) g(\boldsymbol{k}),$$
   and argue that if $g(\boldsymbol{k}) \neq 0$ only near the Fermi surface, one obtains the equation shown in the lectures.

2. **Bosonic and fermionic atoms**
   Particles with integer and half-integer total spin follow boson and fermion statistics, respectively. By calculating the numbers of protons, neutrons and electrons (which all have spin 1/2), deduce which of the following particles are bosons and which of them are fermions:
$$^{1}\text{H},\ ^{3}\text{He},\ ^{4}\text{He},\ ^{6}\text{Li},\ ^{7}\text{Li},\ ^{23}\text{Na},\ ^{87}\text{Rb}.$$

3. **Bose condensation and Fermi temperatures**
   In the course of statistical physics it is shown, that the Bose condensation (for spin=0 particles) occurs below the temperature
$$T_{\text{BOSE}} = 3.31\frac{\hbar^2}{mk_B}\left(\frac{N}{V}\right)^{2/3},$$
   where m is the mass of the boson and N/V the number density. Using the mass densities of ^{3}He and ^{4}He ($\rho_3 = 0.081$ g/cm^3 and $\rho_4 = 0.145$ g/cm^3) calculate the Fermi temperature for ^{3}He and the Bose condensation temperature for ^{4}He, and compare them with the superfluid transition temperatures $T_3 = 0.93$ mK and $T_4 = 2.17$ K observed for the two helium isotopes.

4. **Cooper problem**
   Check all the intermediate steps in the Cooper problem shown in the lecture notes.

   (Hint: Orthonormality of plane waves $\int e^{i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{r}}d^3r = L^3\delta_{\boldsymbol{k},\boldsymbol{q}}$.)

5. **Repulsive Cooper problem**
   Think of Cooper's problem for a repulsive interaction, i.e. $g < 0$ in the notation of the lecture notes. Where does the calculation differ from the case of an attractive interaction, $g > 0$? In both cases you can look at the weak-coupling limit $|g|N(0) \ll 1$.

   [Hint: Solve
$$\frac{1}{N(0)g} = \frac{1}{2}\ln\frac{2\epsilon_F - E + 2\epsilon_c}{2\epsilon_F - E}. \tag{2}$$
   exactly before making assumptions on the sign or magnitude of $gN(0)$.]

1. **Fermion creation and annihilation opertators**
The second quantization for fermions is based on the operator a and its hermitian conjugate $a^\dagger$. The only requirements on them are the anticommutator rules ($\{A, B\} = AB + BA$)
$$\{a, a^\dagger\} = 1, \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0. \tag{3}$$
   (i) Show that $a^\dagger a^\dagger = aa = 0$.
   (ii) Prove the relation $a^\dagger a(1 - a^\dagger a) = 0$, which implies that the eigenvalues of the operator $a^\dagger a$ must be 0 and 1. Let $|0\rangle$ and $|1\rangle$ denote the eigenstates.
   (iii) Prove the relations $[a^\dagger a, a] = -a$ and $[a^\dagger a, a^\dagger] = a^\dagger$. Applying these on $|0\rangle$ and $|1\rangle$, show that (with appropriate definitions of phases)
$$a^\dagger |0\rangle = |1\rangle, \quad a|1\rangle = |0\rangle, \quad a^\dagger |1\rangle = 0, \quad a|0\rangle = 0. \tag{4}$$
Note the important difference between 0 and $|0\rangle$.

2. **Fermion many-body states**
Consider a fermion system in occupation-number representation, where the basis vectors have the form
$$|n_1, n_2, n_3, ..., n_\infty\rangle . \tag{5}$$
Here $1, 2, ...$ label the levels, and $n_1, \ n_2, \dots$ are their occupations (= 0 or 1). Second quantization means simply a new notation, where the approach of the previous exercise is applied to each level separately, i.e. for each level there are separate a_i and $a_i^\dagger$'s. The previous anticommutator rules are generalized trivially:
$$\{a_r, a_s^\dagger\} = \delta_{rs}, \quad \{a_r, a_s\} = \{a_r^\dagger, a_s^\dagger\} = 0. \tag{6}$$
   (a) Show that the number operators for states i and j commute: $[a_i^\dagger a_i, a_j^\dagger a_j] = 0$.

Because of the anticommutation of the operators on different levels, one has to be careful in defining the signs in Eq. (5) correctly. One consistent way is to define
$$|n_1, n_2, ..., n_\infty\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}...(a_\infty^\dagger)^{n_\infty} |0, 0, ..., 0\rangle , \tag{7}$$
where the levels always appear in the same (but otherwise arbitrary) order. Correspondingly we write
$$\langle n_1, n_2, ..., n_\infty| = \langle 0, 0, ..., 0| \, a_\infty^{n_\infty} \dots a_2^{n_2} a_1^{n_1} \tag{8}$$
because $(AB)^\dagger = B^\dagger A^\dagger$. We assume the vacuum states normalized, $\langle 0, 0, \dots |0, 0, \dots\rangle = 1$.
   (b) Prove that the states $|\{n_k\}\rangle \equiv |n_1, n_1, \dots, n_\infty\rangle$ are orthonormal: $\langle\{n_k\}|\{n_k'\}\rangle = \delta_{n_1,n_1'}\delta_{n_2,n_2'} \cdots$.

(c) Show that $\langle \{n_k\}|a_i^\dagger a_j|\{n_k\}\rangle = \delta_{ij} n_i$.

(d) Express the state $a_u a_t|0,0,...,1_k,...,1_l,...\rangle$ in a simpler form. Here k and l label two levels and all the occupation numbers not shown are zero.

3. **Fermion Hamiltonian**
Consider now the usual Hamiltonian in the "first" quantization $H = \sum_{k=1}^N T(\boldsymbol{r}_k) + \frac{1}{2}\sum_{k\neq l=1}^N V(\boldsymbol{r}_k,\boldsymbol{r}_l)$, where N is the number of particles, $T(\boldsymbol{r}_k) = -\hbar^2\nabla^2_{\boldsymbol{r}_k}/2m$ is the kinetic energy operator and $V(\boldsymbol{r}_k,\boldsymbol{r}_l)$ the interaction potential. In the "second" quantization, this operator transforms to

$$\hat{H} = \sum_{rs} a_r^\dagger T_{rs} a_s + \frac{1}{2}\sum_{rstu} a_r^\dagger a_s^\dagger V_{rs,tu} a_u a_t, \tag{9}$$

where T_{rs} and $V_{rs,tu}$ are the matrix elements of $T(\boldsymbol{r}_k)$ and $V(\boldsymbol{r}_k,\boldsymbol{r}_l)$ (in $V_{rs,tu}$ states t and u are the initial states, and r and t refer to the same particle). Without trying to go through the lengthy and dull derivation of Eq. (9) (see for example, Fetter-Walecka, Quantum Theory of Many-Particle Systems, pages 3-18), use the equations of previous problem to calculate the following matrix elements and interpret the results:

$$\begin{aligned}
&\langle 0,0,...|\hat{H}|0,0,...\rangle\\
&\langle 0,0,...,1_k,...,1_l,...|\hat{H}|0,0,...\rangle\\
&\langle 0,0,...,1_k,...|\hat{H}|0,0,...,1_l,...\rangle\\
&\langle 0,0,...,1_k,...,1_l,...|\hat{H}|0,0,...,1_k,...,1_l,...\rangle\\
&\langle 0,0,...,1_k,...,1_l,...|\hat{H}|0,0,...,1_v,...,1_w,...\rangle.
\end{aligned}$$

Here all the occupation numbers not shown are zero, and k, l, v and w refer to different levels. (Notice that the exchange terms are automatically included.)

4. **Fourier transforms**
When using the "box-normalized" (L-periodic) plane waves $\phi_{\boldsymbol{k}}(\boldsymbol{r}) = (1/\sqrt{L^3})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ as a basis, the following definitions for the Fourier transformation and its inverse are convenient:

$$F(\boldsymbol{k}) = \int d^3r e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{r}), \qquad f(\boldsymbol{r}) = \frac{1}{L^3}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} F(\boldsymbol{k}).$$

Check by substitution of one into the other that they are consistent with each other if

$$\frac{1}{L^3}\int d^3r e^{\pm i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = \delta_{\boldsymbol{k},\boldsymbol{k}'}, \qquad \frac{1}{L^3}\sum_{\boldsymbol{k}} e^{\pm i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} = \delta(\boldsymbol{r}-\boldsymbol{r}').$$

These are the "orthonormality" and "completeness" relations. Above, all $\boldsymbol{r}$ integrals are over a cube with sides of length L, and the $\boldsymbol{k}$ sums are over the discrete values $\boldsymbol{k} = \frac{2\pi}{L}(n_x,n_y,n_z)$, where $n_{x,y,z}$ are integers. Calculate the orthonormality integral explicitly at least in the case of one dimension.

1. **Fermion Hamiltonian in plane wave basis**
The matrix elements including spin variables are defined

$$\langle \boldsymbol{k}_1\lambda_1|T|\boldsymbol{k}_2\lambda_2\rangle = \sum_\sigma \int d^3r\, \phi^*_{\boldsymbol{k}_1\lambda_1}(\boldsymbol{r},\sigma)\left(-\frac{\hbar^2}{2m}\nabla^2\right)\phi_{\boldsymbol{k}_2\lambda_2}(\boldsymbol{r},\sigma),$$

$$\langle \boldsymbol{k}_1\lambda_1\boldsymbol{k}_2\lambda_2|V|\boldsymbol{k}_3\lambda_3\boldsymbol{k}_4\lambda_4\rangle$$
$$= \sum_\sigma\sum_{\sigma'}\int d^3r\int d^3r'\, \phi^*_{\boldsymbol{k}_1\lambda_1}(\boldsymbol{r},\sigma)\phi^*_{\boldsymbol{k}_2\lambda_2}(\boldsymbol{r}',\sigma')V(\boldsymbol{r}-\boldsymbol{r}')\phi_{\boldsymbol{k}_3\lambda_3}(\boldsymbol{r},\sigma)\phi_{\boldsymbol{k}_4\lambda_4}(\boldsymbol{r}',\sigma').$$

[Here we have assumed a spin-independent translation-invariant potential $V(\boldsymbol{r},\boldsymbol{r}') = V(\boldsymbol{r}-\boldsymbol{r}')$.] Calculate the matrix elements using plane waves

$$\phi_{\boldsymbol{k}\lambda}(\boldsymbol{r},\sigma) = \frac{1}{L^{3/2}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\delta_{\lambda\sigma}. \tag{10}$$

Show that the general Hamiltonian given in the lecture notes reduces to

$$\check{H} = \sum_{\boldsymbol{k},\sigma}\epsilon_k \check{a}^\dagger_{\boldsymbol{k}\sigma}\check{a}_{\boldsymbol{k}\sigma} + \frac{1}{2L^3}\sum_{\boldsymbol{k}_1,\sigma}\sum_{\boldsymbol{k}_2,\lambda}\sum_{\boldsymbol{k}_3}\sum_{\boldsymbol{k}_4} V(\boldsymbol{k}_1-\boldsymbol{k}_3)\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4}\check{a}^\dagger_{\boldsymbol{k}_1\sigma}\check{a}^\dagger_{\boldsymbol{k}_2\lambda}\check{a}_{\boldsymbol{k}_4\lambda}\check{a}_{\boldsymbol{k}_3\sigma}.$$

Here σ, $\lambda = \uparrow$ or $\downarrow$ and $V(\boldsymbol{k}) = \int d^3r V(\boldsymbol{r})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$.

(Hint: Change to coordinates $\tilde{\boldsymbol{r}} = \boldsymbol{r}-\boldsymbol{r}'$ and $\boldsymbol{R} = (\boldsymbol{r}+\boldsymbol{r}')/2$ in the double integral: $\int d^3r\int d^3r' = \int d^3R\int d^3\tilde{r}$.)

2. **Contact interaction**
As a special case of the previous exercise, consider a contact interaction $V(\boldsymbol{r}) = -g\delta(\boldsymbol{r})$. Write the Hamiltonian in this case. Show that terms with $\sigma = \lambda$ vanish, and those with $\sigma \neq \lambda$ are equal, as mentioned in the lecture notes.

3. **Bogoliubov transformation**
Do all the intermediate steps of the Bogoliubov transformation [Eqs. (148)-(156)] not shown in the lecture notes. That is, check carefully the result $\check{K}_{\text{eff}} = \sum_{\boldsymbol{k}\sigma} E_k\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma} + \Omega_0$. What is Ω_0? You can save yourself some work by not following precisely the route implied in the lecture notes, but rather starting from the matrix form (147), and inserting

$$\begin{pmatrix}\check{a}_{\boldsymbol{k}\uparrow}\\ \check{a}^\dagger_{-\boldsymbol{k}\downarrow}\end{pmatrix} = U_k\begin{pmatrix}\check{\gamma}_{\boldsymbol{k}\uparrow}\\ \check{\gamma}^\dagger_{-\boldsymbol{k}\downarrow}\end{pmatrix} = \begin{pmatrix}u_k^* & v_k\\ -v_k^* & u_k\end{pmatrix}\begin{pmatrix}\check{\gamma}_{\boldsymbol{k}\uparrow}\\ \check{\gamma}^\dagger_{-\boldsymbol{k}\downarrow}\end{pmatrix}.$$

Require the coefficient matrix U_k to be unitary ($U_kU_k^\dagger = U_k^\dagger U_k = 1$), and then require that $U_k^\dagger K_k U_k$ is diagonal, where K_k is the Hermitian matrix appearing in the Hamiltonian. These two requirements give you the equations for u_k and v_k [(150) and (152)]. When solving them, you can assume Δ to be real, as in the lecture notes.

1. **Grand potential for a superconductor**
Calculate the grand potential $\Omega = -\ln[\mathrm{Tr}\, e^{-\beta\check{K}}]/\beta$ for $\check{K}_{\text{eff}} = \sum_{\boldsymbol{k},\sigma} E_k \check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma} + \Omega_0$ to obtain the result
$$\Omega = \Omega_0 - \frac{2}{\beta}\sum_{\boldsymbol{k}} \ln\left(1 + e^{-\beta E_k}\right).$$
(Hint: Calculate the trace $\mathrm{Tr}[\cdots] = \sum_{\{n_{\boldsymbol{k}\sigma}\}=0}^{1} \langle\{n_{\boldsymbol{k}\sigma}\}|\cdots|\{n_{\boldsymbol{k}\sigma}\}\rangle$ in the basis of the γ number operator $\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma}|\{n_{\boldsymbol{k}'\sigma'}\}\rangle = n_{\boldsymbol{k}\sigma}|\{n_{\boldsymbol{k}'\sigma'}\}\rangle$.)

2. **Gap equation: derivation**
Do all the intermediate steps in deriving the gap equation [(163)] from the definition
$$\Delta = \frac{g}{L^3}\sum_{\boldsymbol{k}} \langle \check{a}_{-\boldsymbol{k}\downarrow}\check{a}_{\boldsymbol{k}\uparrow}\rangle.$$
(Hint: Make use of the previous exercise to show $\langle\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma}\rangle = n(E_k)$, where $n(E) = 1/(e^{\beta E}+1)$, and $\langle\check{\gamma}_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}'\sigma'}\rangle = \langle\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}^\dagger_{\boldsymbol{k}'\sigma'}\rangle = 0$.)

3. **Hartree-Fock interaction**
Show that the Hartree-Fock (not anomalous) interaction energy $\langle\check{V}_{\text{HF}}\rangle$ is the same for normal and superconducting states. This demonstrates *a posteriori* that the neglect of non-anomalous HF terms in the treatment of the superconducting state is allowed. The HF potential energy for a spin-conserving contact interaction can be written as
$$\check{V}_{\text{HF}} = \frac{g}{L^3}\sum_{\boldsymbol{k}_1}\sum_{\boldsymbol{k}_2}\sum_{\boldsymbol{k}_3}\sum_{\boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4}\Big(\langle\check{a}^\dagger_{\boldsymbol{k}_3\uparrow}\check{a}_{\boldsymbol{k}_1\uparrow}\rangle\check{a}^\dagger_{\boldsymbol{k}_4\downarrow}\check{a}_{\boldsymbol{k}_2\downarrow} + \langle\check{a}^\dagger_{\boldsymbol{k}_4\downarrow}\check{a}_{\boldsymbol{k}_2\downarrow}\rangle\check{a}^\dagger_{\boldsymbol{k}_3\uparrow}\check{a}_{\boldsymbol{k}_1\uparrow} - \langle\check{a}^\dagger_{\boldsymbol{k}_4\downarrow}\check{a}_{\boldsymbol{k}_2\downarrow}\rangle\langle\check{a}^\dagger_{\boldsymbol{k}_3\uparrow}\check{a}_{\boldsymbol{k}_1\uparrow}\rangle\Big).$$
(Hint: as intermediate results show that $\langle\check{a}^\dagger_{\boldsymbol{k}_3\uparrow}\check{a}_{\boldsymbol{k}_1\uparrow}\rangle = \delta_{\boldsymbol{k}_3,\boldsymbol{k}_1}C_{k_1}$ and $\langle\check{V}_{\text{HF}}\rangle = (g/L^3)(\sum_{\boldsymbol{k}} C_k)^2$ and deduce that the sum for $|\xi_k| < \epsilon_c$ is independent of Δ.)

4. **Gap equation at $T = T_c$**
Prove the result
$$\int_0^{\epsilon_c} d\xi\,\frac{1}{\xi}\tanh\frac{\xi}{2k_BT_c} = \ln\frac{2\epsilon_c e^\gamma}{\pi k_B T_c},$$
where γ is a certain constant given by the definite integral:
$$\int_0^\infty \mathrm{d}x\,\frac{\ln x}{\cosh^2 x} = -\gamma + \ln\frac{\pi}{4}.$$
(Hint: use $\xi^{-1} = d(\ln\xi)/d\xi$ and integrate by parts. Verify that the integral converges, so that you can take the limit $\epsilon_c \to \infty$.)

5. **Gap equation: elimination of $gN(0)$ and ϵ_c**
Show that the gap equation (in weak coupling approximation $gN(0) \ll 1$) can be written in the form

$$\ln \frac{T_c}{T} = \int_0^\infty \left(\frac{\tanh(\xi/2k_B T)}{\xi} - \frac{\tanh(\sqrt{\xi^2+\Delta^2}/2k_B T)}{\sqrt{\xi^2+\Delta^2}} \right) d\xi.$$

Verify that the integral converges, so that it is possible to put $\epsilon_c \to \infty$. In this way the two parameters $gN(0)$ and ϵ_c have been replaced by a single one: T_c.

1. **BCS ground state**
   Show that $\check{\gamma}_{\boldsymbol{k}\sigma}|\psi_0\rangle = 0$ for the BCS ground state $|\psi_0\rangle$, which means that $|\psi_0\rangle$ is the vacuum state for excitations. Consider at least the case $\sigma = \uparrow$.

   (Hint: It is useful to define $c_{\boldsymbol{k}} = u_k + v_k a^\dagger_{\boldsymbol{k}\uparrow} a^\dagger_{-\boldsymbol{k}\downarrow}$ and to show that $[c_{\boldsymbol{k}}, c_{\boldsymbol{k}'}] = 0$.)

2. **Normalization of the BCS ground state**
   Assuming that $\langle \mathrm{vac}|\mathrm{vac}\rangle = 1$, show that the BCS ground state $|\psi_0\rangle$ is normalized as $\langle\psi_0|\psi_0\rangle = 1$.

3. **Excitations of BCS state**
   Let $|\psi_0\rangle$ be the BCS ground state. Show that the excited states $\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}|\psi_0\rangle$ are of the form where the single-particle state $\boldsymbol{k}\sigma$ (to which particles are created by $\check{a}^\dagger_{\boldsymbol{k}\sigma}$) is populated and $-\boldsymbol{k}-\sigma$ is empty. You can limit to the case $\sigma = \uparrow$.

4. **Energy functional**
   Show that for the energy functional ($u_k^2 = 1 - v_k^2$, $\xi_k = \hbar^2 k^2/2m - \mu$)

$$\begin{aligned}\Omega(T, V, \mu, v_k, \Delta) =& 2\sum_{\boldsymbol{k}}(\xi_k v_k^2 - \Delta u_k v_k) + \frac{L^3}{g}\Delta^2 \\ & - 2k_B T \sum_{\boldsymbol{k}} \ln(1 + e^{-\sqrt{\xi_k^2+\Delta^2}/k_B T}),\end{aligned}$$

   the relations

$$\frac{\partial\Omega}{\partial v_k} = 0, \qquad \frac{\partial\Omega}{\partial\Delta} = 0,$$

   are equivalent with the conditions (152) and (158) of the lecture notes.

1. **Superconducting ground state energy**
   Derive the $T = 0$ relations shown in the lectures
   $$\Omega_0 - \Omega_0(\Delta = 0) = -\frac{1}{2}\sum_{\mathbf{k}} \frac{\Delta^4}{E_k(E_k + |\xi_k|)^2},$$
   and
   $$\Omega_0 - \Omega_0(\Delta = 0) = -\frac{1}{2}L^3 N(0)\Delta^2.$$
   (Hint: Apply the gap equation and use $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2+1}(\sqrt{x^2+1}+|x|)^2} = 1$.)

2. **Specific heat**
   Show that from the grand potential $\Omega(T, V, \mu) = \min_{v_k,\Delta}[\Omega(T, V, \mu, v_k, \Delta)]$ one obtains the specific heat
   $$C = \frac{L^3 N(0)}{2k_B T^2} \int_{-\infty}^{\infty} d\xi \frac{1}{\cosh^2 \frac{\sqrt{\xi^2+\Delta^2}}{2k_B T}} \left(\xi^2 + \Delta^2 - T\Delta \frac{d\Delta}{dT}\right).$$

3. **Normal state specific heat**
   Show that for the specific heat of the normal state one obtains
   $$C = \frac{2\pi^2}{3} L^3 N(0) k_B^2 T.$$
   (Hint: Use $\int_{-\infty}^{\infty} \frac{x^2}{\cosh^2 x} dx = \frac{\pi^2}{6}$.)

4. **Variation of Ginzburg-Landau energy functional**
   A variation $\psi^* \to \psi^* + \delta\psi^*$ or $\boldsymbol{A} \to \boldsymbol{A} + \delta\boldsymbol{A}$, for example, changes the GL energy functional according to $G \to G + \delta G$. Derive the GL differential equations and their boundary conditions by requiring $\delta G = 0$ to lowest order in arbitrary variations.

   (Hint: You may first want to prove the more general result that the minimum conditions for $G[\psi^*, \boldsymbol{A}] = \int d^3r g(\psi^*, \nabla\psi^*, \boldsymbol{A}, \nabla \times \boldsymbol{A})$ are $\frac{\partial g}{\partial \psi^*} - \nabla \cdot \frac{\partial g}{\partial \nabla\psi^*} = 0$, $\frac{\partial g}{\partial \boldsymbol{A}} + \nabla \times \frac{\partial g}{\partial \nabla\times\boldsymbol{A}} = 0$, $\hat{\boldsymbol{n}} \cdot \frac{\partial g}{\partial \nabla\psi^*} = 0$, $\hat{\boldsymbol{n}} \times \frac{\partial g}{\partial \nabla\times\boldsymbol{A}} = 0$. Use relations like $\boldsymbol{C} \cdot \nabla\phi = \nabla \cdot (\boldsymbol{C}\phi) - (\nabla \cdot \boldsymbol{C})\phi$.)

1. **Boundary conditions of GL theory**
   Show by using the GL differential equations that in GL theory the continuity equation $\boldsymbol{\nabla}\cdot\boldsymbol{j}=0$ and the boundary condition $\hat{\boldsymbol{n}}\cdot\boldsymbol{j}=0$ are satisfied, where $\boldsymbol{j}$ is the current density and $\hat{\boldsymbol{n}}$ is the surface normal.

2. **Surface current in applied magnetic field**
   In the lecture notes the behavior of the magnetic field was derived in the case of a superconducting half space, when an external field is applied parallel to the surface. Calculate the related vector potential, current density and the total current. (Hint: If $\boldsymbol{B}=B(x)\hat{\boldsymbol{z}}$, you can assume that $\boldsymbol{A}=A(x)\hat{\boldsymbol{y}}$.)

3. **Dimensionless GL theory**
   Show that by choosing units of length, energy, order parameter, and magnetic field properly, and neglecting constant energy terms, the GL free energy functional in a given external field $\boldsymbol{B}_{\text{ext}}=\boldsymbol{\nabla}\times\boldsymbol{A}_{\text{ext}}$ at $T<T_c$ can be written in the dimensionless form

$$G(\Psi,\boldsymbol{A})=\int d^3x\left[-|\Psi|^2+\frac{1}{2}|\Psi|^4+|(\boldsymbol{\nabla}+\mathrm{i}\boldsymbol{A})\Psi|^2+\kappa^2|\boldsymbol{\nabla}\times(\boldsymbol{A}-\boldsymbol{A}_{\text{ext}})|^2\right],$$

   which contains only one dimensionless parameter $\kappa=\lambda(T)/\xi_{GL}(T)$.

4. **GL equation in 1 D**
   Consider the one-dimensonal GL equation

$$\xi_{GL}^2\frac{d^2f}{dx^2}+f-f^3=0.$$

   Its first integral can be derived by analogy, by comparing the GL energy to the action integral $S=\int dtL(\{\dot{q}_i\},\{q_i\},t)$ where $L=T-V$, and noting that when $\partial L/\partial t=0$, the Hamiltonian $H=\sum_i\dot{q}_i\frac{\partial L}{\partial\dot{q}_i}-L$ is constant due to the equation of motion $\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}-\frac{\partial L}{\partial q_i}=0$ (which in this analogy is the GL equation). Do this. Show that

$$f(x)=\tanh\frac{x}{\sqrt{2}\xi_{GL}}$$

   satisfies the first-integral equation. You can find the first integral also without the analogy and solve it for f directly, if you prefer that.

5. **Specific heat discontinuity**
   Calculate the discontinuity ΔC of the specific heat at $T=T_c$ in the G-L theory. Using the microscopic (BCS) values for the G-L parameters and the known result for the specific heat $C_n(T)$ of the normal state, show that

$$\frac{\Delta C}{C_n(T_c)}=\frac{12}{7\zeta(3)}=1.43.$$

   (Hint: The specific heat is, as usual $C=-T(\partial^2G/\partial T^2)$.)

1. **Critical current in a wire**
   Consider G-L equations in a thin wire, assuming that $\boldsymbol{A} = 0$ and
   $$\Psi(x) = Ce^{ikx}.$$
   Calculate the supercurrent j and the G-L energy for this state. Minimize the energy with respect to C at constant k. Describe C and j as functions of k. Find the maximum supercurrent and the corresponding k.

2. **Normal-superconductor interface:** $\lambda \ll \xi_{\text{GL}}$
   Calculate the energy of an interface between normal and superonducting states (in the critical field H_c) in the limit $\kappa \to 0$, in which case the you can neglect the magnetic field on the superconducting side and use the solution $f(x) = \tanh\left(x/\sqrt{2}\xi_{GL}\right)$. Note that the free energy densities of the normal and superconducting states (at $x = \pm\infty$) have to be the same in order to have a stable interface.

   (Hint: In this limit the N-S interface is abrupt, and you can choose it to be at $x = 0$ for example. At this point f is continuous. On the N side $B = \mu_0 H_c$ and $f = 0$.)

3. **Vortex density**
   Determine the density of vortices (number per cross-sectional area) in a rotating superfuid by starting from the assumption that the velocity on the edge of the cylindrical container is on average the same as the velocity of the edge. How many vortices are there in a cylinder of radius 5 mm that makes one revolution per second? Consider separately ^{3}He and ^{4}He.

   (Hint: The circulation $\oint d\boldsymbol{l} \cdot \boldsymbol{v}_s$ around N vortices is $N\frac{h}{m}$.)

4. **Rotating superconductor**
   In a superconductor one can define the velocity of the superconducting part as
   $$\boldsymbol{v}_s = \frac{1}{m}\left(\hbar\boldsymbol{\nabla}\chi - q\boldsymbol{A}\right).$$
   When rotating the superconductor, no vortices are generated, but a uniform magnetic field is. Calculate it. You can assume the condensate to rotate as a solid body together with the atomic lattice.

1. **Flux line density**
   Verify that the dimension of length is in accordance with the given field of 1 Tesla in the experimental vortex lattice picture given in the lecture notes.

2. **Level density in a superconductor**
   Show that the density of the single-particle energies $E_k = \sqrt{\xi_k^2 + \Delta^2}$ (i.e. density of levels) in a superconductor is 0 for $0 < E < \Delta$ and

$$N_s(E) = \frac{N_n(0)E}{\sqrt{E^2 - \Delta^2}}$$

   for $E > \Delta$, where $N_n(0)$ is the corresponding normal-state ($\Delta = 0$) density of levels.

3. **Josephson current**
   Show, as instructed in the lecture notes, that if the Jospehson coupling energy is $F_J(\Delta\phi) = -E_J \cos\Delta\phi$, then the Josephson current is $J = \frac{|q|}{\hbar} E_J \sin\Delta\phi$. Thus consider for simplicity a quasi-one-dimensional model, with a first superconductor at $-L < x < 0$ connected to a second one at $0 < x < L$ via a tunnel barrier at $x = 0$. The energy is of the form $F = S\int_{-L}^{0} dx f(x) + S\int_0^L dx f(x) + F_J$, where S is a cross-sectional area, $f(x)$ is the GL energy density at zero magnetic field, with $\psi(x) = \psi_0 e^{i\phi(x)}$, where ψ_0 is a real constant, and $\Delta\phi = \phi(0^-) - \phi(0^+)$. By considering variations $\phi(x) \to \phi(x) + \delta\phi(x)$ where $\delta\phi(\pm L) = 0$, show that the equilibrium conditions arising from the surface terms of $\delta F = 0$ at $x = 0^\pm$ imply that the current $J = Sj(0^-) = Sj(0^+)$ satisfies $J = \frac{|q|}{\hbar}\partial F_J(\Delta\phi)/\partial\Delta\phi$.

4. **DC SQUID**
   Starting from the equations in the lectures

$$\begin{aligned}\Delta\phi_1 + \Delta\phi_2 &= \frac{2\pi\Phi}{\Phi_0} + 2\pi N \\ J &= J_{c1}\sin(\Delta\phi_1) - J_{c2}\sin(\Delta\phi_2), \end{aligned} \tag{11}$$

   show that for $J_{c1} = J_{c2} = J_c$ the current can be written

$$J = 2J_c(-1)^N \cos\frac{\pi\Phi}{\Phi_0}\sin\frac{\Delta\phi_1 - \Delta\phi_2}{2}. \tag{12}$$