1. Using the approach given in lecture notes, derive the equations

$$\Omega = E - \mu N - ST, \qquad \Omega = -\frac{1}{\beta}\ln[\mathrm{Tr}(e^{-\beta(\hat{H}-\mu\hat{N})})], \qquad d\Omega = -SdT - pdV - Nd\mu$$

Solution: The probability density operator in the presence of variable particle number is $\hat{\rho} = e^{\beta(\Omega - \hat{H} + \mu\hat{N})}$. Now:

i)

$$\begin{aligned}
\hat{\rho} &= e^{\beta(\Omega - \hat{H} + \mu\hat{N})} \\
\ln\hat{\rho} &= \beta(\Omega + \hat{H} + \mu\hat{N}) \\
\Omega &= \hat{H} - \mu\hat{N} + k_B T \ln\hat{\rho} \\
\langle\Omega\rangle &= \langle\hat{H}\rangle - \mu\langle\hat{N}\rangle - T(-k_B\langle\ln\hat{\rho}\rangle) \\
\Omega &= E - \mu N - TS
\end{aligned}$$

ii)

$$\begin{aligned}
\hat{\rho} &= e^{\beta(\Omega - \hat{H} + \mu\hat{N})} \\
\hat{\rho} &= e^{\beta\Omega} e^{-\beta(\hat{H} - \mu\hat{N})} \\
e^{-\beta\Omega}\hat{\rho} &= e^{-\beta(\hat{H} - \mu\hat{N})} \\
e^{-\beta\Omega}\underbrace{\mathrm{Tr}\hat{\rho}}_{=1} &= \mathrm{Tr}e^{-\beta(\hat{H} - \mu\hat{N})} \\
e^{-\beta\Omega} &= \mathrm{Tr}e^{-\beta(\hat{H} - \mu\hat{N})} \\
-\beta\Omega &= \ln[\mathrm{Tr}e^{-\beta(\hat{H} - \mu\hat{N})}] \\
\Omega &= -\frac{1}{\beta}\ln[\mathrm{Tr}e^{-\beta(\hat{H} - \mu\hat{N})}]
\end{aligned}$$

iii)

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}) &= 1, \\
d(\mathrm{Tr}\hat{\rho}) &= 0 \\
\mathrm{Tr}(d\hat{\rho}) &= 0
\end{aligned}$$

$$\mathrm{Tr}\left\{\beta\left[d\Omega-\frac{d\hat{H}}{d\lambda}d\lambda+\hat{N}d\mu+\mu d\hat{N}-(\Omega-\hat{H}+\mu\hat{N})\frac{dT}{T}\right]\hat{\rho}\right\}=0$$

$$\beta\left[d\Omega\underbrace{\mathrm{Tr}\hat{\rho}}_{=1}-\underbrace{\mathrm{Tr}(\frac{d\hat{H}}{d\lambda}\hat{\rho})}_{=-p}\underbrace{d\lambda}_{=dV}+\underbrace{\mathrm{Tr}(\hat{N}\hat{\rho})}_{\langle\hat{N}\rangle=N}d\mu+\mu\underbrace{\mathrm{Tr}(\hat{\rho}d\hat{N})}_{=0}-\underbrace{\mathrm{Tr}[(\Omega-\hat{H}+\mu\hat{N})\hat{\rho}]}_{=\Omega-E+\mu N=-ST}\frac{dT}{T}\right]=0$$

$$\beta[d\Omega+pdV+Nd\mu+SdT]=0$$

Thus for equilibrium states $d\Omega=-pdV-Nd\mu-SdT$.

Note: Above we used $\mathrm{Tr}[(\Omega-\hat{H}+\mu\hat{N})\hat{\rho}]=k_B\mathrm{Tr}[\hat{\rho}\ln\hat{\rho}]T=-ST$. We also set $\mathrm{Tr}(\hat{\rho}d\hat{N})=0$, because while $\langle\hat{N}\rangle$ can depend on $\lambda=V$ via $\hat{\rho}$ (since $\hat{H}$ does), the operator $\hat{N}$ itself does not. (This argument is not very satisfying, though.)

2. Calculate Ω for an ideal Bose-gas similarly as it is calculated for an ideal Fermi-gas in the lecture notes. Show that Bose and Fermi distributions reduce to a classical Maxwell-Boltzmann distribution

$$f(\epsilon)=e^{-\beta(\epsilon-\mu)},\qquad \text{when}\qquad \beta(\epsilon-\mu)\gg 1.$$

(Hint: The only difference in the calculation is that for bosons $n_\alpha=0,1,2,\ldots$ instead of $n_\alpha=0,1$.)

Solution: The grand potential Ω for bosons is calculated similarly as for fermions, starting from equation (30) in the lectures. The only difference is that $n_i=0,1,2,\ldots$ So, using $(\hat{H}-\mu\hat{N})|n_1n_2\ldots\rangle=(\sum_j\epsilon_jn_j-\mu\sum_jn_j)|n_1n_2\ldots\rangle$, we have

$$\begin{aligned}
e^{-\beta\Omega}=\mathrm{Tr}e^{-\beta(\hat{H}-\mu\hat{N})}&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\langle n_1n_2\ldots|e^{-\beta(\hat{H}-\mu\hat{N})}|n_1n_2\ldots\rangle\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\langle n_1n_2\ldots|e^{-\beta\sum_j(\epsilon_j-\mu)n_j}|n_1n_2\ldots\rangle\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots e^{-\beta(\epsilon_1-\mu)n_1}e^{-\beta(\epsilon_2-\mu)n_2}\ldots\\
&=\sum_{n_1=0}^{\infty}e^{-\beta(\epsilon_1-\mu)n_1}\sum_{n_2=0}^{\infty}e^{-\beta(\epsilon_2-\mu)n_2}\cdots=\prod_\alpha\sum_{n_\alpha=0}^{\infty}[e^{-\beta(\epsilon_\alpha-\mu)}]^{n_\alpha}
\end{aligned}$$

For a geometric series we have $\sum_{j=0}^{\infty}q^j=1/(1-q)$, if $|q|<1$. For Bose gas $\epsilon-\mu>0$ and so the sum converges. Thus

$$\begin{aligned}
e^{-\beta\Omega}&=\prod_\alpha\frac{1}{1-e^{-\beta(\epsilon_\alpha-\mu)}}\\
\Omega=-\frac{1}{\beta}\ln\left[\prod_\alpha\frac{1}{1-e^{-\beta(\epsilon_\alpha-\mu)}}\right]&=\frac{1}{\beta}\sum_\alpha\ln\left[1-e^{-\beta(\epsilon_\alpha-\mu)}\right]
\end{aligned}$$

and

$$N = -\left(\frac{d\Omega}{d\mu}\right)_{T,V} = -\frac{1}{\beta}\sum_{\alpha}\frac{-\beta e^{-\beta(\epsilon_\alpha-\mu)}}{1-e^{-\beta(\epsilon_\alpha-\mu)}} = \sum_{\alpha}\frac{1}{e^{\beta(\epsilon_\alpha-\mu)}-1}$$

So identifying $N = \sum_\alpha f_\alpha$, we have

$$f_\alpha = \frac{1}{e^{\beta(\epsilon_\alpha-\mu)}-1}$$

which is the Bose-Einstein distribution. Now consider the limit $\beta(\epsilon_\alpha - \mu) \gg 1$. Clearly then $e^{\beta(\epsilon_\alpha-\mu)} \gg 1$ and thus $f_\alpha \approx e^{-\beta(\epsilon_\alpha-\mu)}$, which is the classical (Maxwell-Boltzmann) distribution Similarly for fermions we had the Fermi-Dirac distribution

$$f_\alpha = \frac{1}{e^{\beta(\epsilon_\alpha-\mu)}+1}$$

Also here, if $\beta(\epsilon_\alpha - \mu) \gg 1$, we have $e^{\beta(\epsilon_\alpha-\mu)} \gg 1$ and the classical result is obtained. Thus in this limit there is no distinction between bosons and fermions.

3. Using the grand potential of an ideal Fermi-gas

$$\Omega = -k_BT\sum_\alpha \ln[1+e^{\beta(\mu-\epsilon_\alpha)}],$$

(a) verify that

$$N = -\frac{\partial\Omega}{\partial\mu} = \sum_\alpha f_\alpha \qquad \text{where} \qquad f_\alpha = \frac{1}{e^{\beta(\epsilon_\alpha-\mu)}+1}.$$

(b) Calculate $S = -\frac{\partial\Omega}{\partial T}$ and show that it can be written in the form

$$S = -k_B\sum_\alpha[f_\alpha \ln f_\alpha + (1-f_\alpha)\ln(1-f_\alpha)].$$

(Hint: You may find this useful: $x = \ln(1-\frac{1}{e^x+1}) - \ln(\frac{1}{e^x+1})$.)

(c) Using $E = \Omega + ST + \mu N$, show that

$$E = \sum_\alpha \epsilon_\alpha f_\alpha.$$

(d) Using $p = -\frac{\partial\Omega}{\partial V}$, show that

$$p = -\sum_\alpha \frac{\partial\epsilon_\alpha}{\partial V} f_\alpha.$$

(e) Show that

$$dS = \frac{1}{T}\sum_\alpha (\epsilon_\alpha - \mu) df_\alpha.$$

(f) Using the previous results show that

$$dE = TdS - pdV + \mu dN.$$

Give an interpretation for the two terms (TdS and $-pdV$) in the special case N =constant.

Solution: The grand potential of an ideal Fermi gas

$$\Omega = -k_B T \sum_\alpha \ln[1 + e^{\beta(\mu - \epsilon_\alpha)}]$$

a)

$$N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V} = +k_B T \sum_\alpha \frac{\beta e^{\beta(\mu-\epsilon_\alpha)}}{1 + e^{\beta(\mu-\epsilon_\alpha)}} = \sum_\alpha \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1} = \sum_\alpha f_\alpha$$

b)

$$\begin{aligned} S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu} &= k_B \sum_\alpha \ln[1 + e^{\beta(\mu-\epsilon_\alpha)}] + k_B T \sum_\alpha \frac{-\beta \frac{1}{T}(\mu - \epsilon_\alpha) e^{\beta(\mu-\epsilon_\alpha)}}{1 + e^{\beta(\mu-\epsilon_\alpha)}} \\ &= -k_B \sum_\alpha \left\{ -\ln[1 + e^{\beta(\mu-\epsilon_\alpha)}] + \frac{\beta(\mu - \epsilon_\alpha)}{e^{\beta(\epsilon_\alpha-\mu)} + 1} \right\} \end{aligned}$$

Some playing around with logarithms and fermi functions is needed. The expression is we must develop is of the form

$$\begin{aligned} -\ln[1 + e^{-x}] - \frac{x}{e^x + 1} &= -\ln[e^{-x}(e^x + 1)] - \frac{x}{e^x + 1} = x + \ln\frac{1}{e^x + 1} - \frac{x}{e^x + 1} \\ &= \ln\frac{1}{e^x + 1} + x\left(1 - \frac{1}{e^x + 1}\right) \end{aligned}$$

Now here

$$x = \ln e^x = \ln\left(\frac{\frac{e^x}{e^x+1}}{\frac{1}{e^x+1}}\right) = \ln\left(\frac{1 - \frac{1}{e^x+1}}{\frac{1}{e^x+1}}\right) = \ln\left(1 - \frac{1}{e^x + 1}\right) - \ln\left(\frac{1}{e^x + 1}\right)$$

and inserting this to the previous expression we find

$$-\ln[1 + e^{-x}] - \frac{x}{e^x + 1} = \frac{1}{e^x + 1}\ln\frac{1}{e^x + 1} + \left(1 - \frac{1}{e^x + 1}\right)\ln\left(1 - \frac{1}{e^x + 1}\right)$$

Applying this with $x = \beta(\epsilon_\alpha - \mu)$ we finally have the desired result

$$S = -k_B \sum_\alpha [f_\alpha \ln f_\alpha + (1 - f_\alpha) \ln(1 - f_\alpha)]$$

where $f_\alpha = 1/(e^{\beta(\epsilon_\alpha - \mu)} + 1)$.

c) From the previous consideration we know that

$$\Omega = -k_B T \ln(1 + e^{\beta(\mu - \epsilon_\alpha)}) = -k_B T \sum_\alpha [-\ln(1 - f_\alpha)] = k_B T \sum_\alpha \ln(1 - f_\alpha)$$

Then

$$\begin{aligned} E &= \Omega + ST + \mu N \\ &= k_B T \sum_\alpha \ln(1 - f_\alpha) - k_B T \sum_\alpha [f_\alpha \ln f_\alpha - f_\alpha \ln(1 - f_\alpha) + \ln(1 - f_\alpha)] + \mu \sum_\alpha f_\alpha \\ &= \sum_\alpha \left[-\frac{1}{\beta} f_\alpha \ln \frac{f_\alpha}{1 - f_\alpha} + \mu f_\alpha \right] = \sum_\alpha \left[-\frac{\beta}{\beta} (\mu - \epsilon_\alpha) f_\alpha + \mu f_\alpha \right] = \sum_\alpha \epsilon_\alpha f_\alpha \end{aligned}$$

d) The only quantities depending on the volume are the single-particle energies ϵ_α. So

$$\begin{aligned} p = -\left(\frac{d\Omega}{dV} \right)_{\mu, T} &= k_B T \sum_\alpha \frac{-\beta \frac{\partial \epsilon_\alpha}{\partial V} e^{\beta(\mu - \epsilon_\alpha)}}{1 + e^{\beta(\mu - \epsilon_\alpha)}} \\ &= -\sum_\alpha \frac{\partial \epsilon_\alpha}{\partial V} \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1} = -\sum_\alpha \frac{\partial \epsilon_\alpha}{\partial V} f_\alpha \end{aligned}$$

e) We just proved

$$S = -k_B \sum_\alpha [f_\alpha \ln f_\alpha + (1 - f_\alpha) \ln(1 - f_\alpha)]$$

So now

$$\begin{aligned} dS &= -k_B \sum_\alpha [df_\alpha \ln f_\alpha + f_\alpha \frac{df_\alpha}{f_\alpha} - df_\alpha \ln(1 - f_\alpha) - (1 - f_\alpha) \frac{df_\alpha}{1 - f_\alpha}] \\ &= -k_B \sum_\alpha df_\alpha \ln \frac{f_\alpha}{1 - f_\alpha} = -k_B \sum_\alpha df_\alpha \beta(\mu - \epsilon_\alpha) = \frac{1}{T} \sum_\alpha (\epsilon_\alpha - \mu) df_\alpha \end{aligned}$$

f) The result $dE = TdS - pdV + \mu dN$ follows generally by differentiating $\Omega = E - \mu N - TS$ and using $d\Omega = -SdT - pdV - Nd\mu$. Using the results of this exercise, we can write

$$\begin{aligned} TdS - pdV + \mu dN &= \sum_\alpha (\epsilon_\alpha - \mu) df_\alpha + \left(\sum_\alpha \frac{\partial \epsilon_\alpha}{\partial V} f_\alpha \right) dV + \mu \sum_\alpha df_\alpha \\ &= \sum_\alpha (\epsilon_\alpha df_\alpha + f_\alpha d\epsilon_\alpha) - \mu \sum_\alpha df_\alpha + \mu \sum_\alpha df_\alpha \\ &= \sum_\alpha (\epsilon_\alpha df_\alpha + f_\alpha d\epsilon_\alpha) \end{aligned}$$

which is indeed equal to $dE = d\left(\sum_\alpha \epsilon_\alpha f_\alpha\right)$. Here we identidfied $d\epsilon_\alpha = \frac{\partial \epsilon_\alpha}{\partial V} dV$. This result now gives a microscopic view to the origin of dE.

The last term, originating here from $-pdV$, describes the effects of "slow" changes. Just the energy levels ϵ_α are shifted, but the occupation probabilities f_α remain unchanged. Such changes can be reversed without dissipation of heat to the environment.

The first term, originating from TdS, describes the effects of "fast" changes. Here the occupation probabilities f_α of the levels (also) change. A change like this is typically irreversible. (The system can be returned to its initial equlibrium state, but not without dissipating heat to the environment.)

Note: The assumption of constant N seems not to be of much relevance. It would allow us to set $dN = \sum_\alpha df_\alpha = 0$ above, but the corresponding terms cancel anyway.

4. Show that an ideal (spin 1/2) Fermi-gas has a pressure

$$p = \frac{2}{3}\frac{E}{V},$$

at all temperatures and at $T = 0$

$$S = 0 \qquad \text{and} \qquad \frac{E}{N} = \frac{3}{5}\epsilon_F.$$

Solution: For ideal Fermi gas the single-particle energy levels have the expression

$$\epsilon_\alpha = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2) = \frac{4\pi^2\hbar^2}{2m}\frac{1}{L^2}\bar{n}_\alpha^2 = \frac{\beta_\alpha}{V^{2/3}}$$

Here we used periodic boundary conditions, giving $k_x = (2\pi/L)n_x$, $n_x = 0, \pm1, \pm2, \ldots$ and so on. We also denoted $(n_x, n_y, n_z) \to \bar{n}_\alpha$ and then $\beta_\alpha = 4\pi^2\hbar^2\bar{n}_\alpha^2/2m$ for brevity. Now

$$p = -\sum_\alpha \frac{\partial \epsilon_\alpha}{\partial V} f_\alpha = -\sum_\alpha \beta_\alpha \frac{-2/3}{V^{5/3}} f_\alpha = \frac{2}{3}\frac{1}{V}\sum_\alpha \frac{\beta_\alpha}{V^{2/3}} f_\alpha = \frac{2}{3}\frac{1}{V}\sum_\alpha \epsilon_\alpha f_\alpha = \frac{2}{3}\frac{E}{V}$$

This result is independent of temperature. As $T \to 0$ all the single-particle states with $\epsilon_\alpha < \epsilon_F$ become filled ($f_\alpha \to 1$), where ϵ_F is the Fermi energy. The states with $\epsilon_\alpha > \epsilon_F$

are emptied ($f_\alpha \to 0$). Now the entropy becomes

$$\begin{aligned}
S &= -k_B \sum_\alpha [f_\alpha \ln f_\alpha + (1-f_\alpha)\ln(1-f_\alpha)] \\
&\to -k_B \Big\{ \sum_{\alpha \text{ filled}} [f_\alpha \ln f_\alpha + (1-f_\alpha)\ln(1-f_\alpha)] \\
&+ \sum_{\alpha \text{ empty}} [f_\alpha \ln f_\alpha + (1-f_\alpha)\ln(1-f_\alpha)] \Big\} \\
&\to -k_B \left[\sum_{\alpha \text{ filled}} (1-f_\alpha)\ln(1-f_\alpha) + \sum_{\alpha \text{ empty}} f_\alpha \ln f_\alpha \right] \\
&\to 0
\end{aligned}$$

Here we used the limit (L'Hôpital rule)

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} x = 0$$

Thus we conclude that at $T = 0$ the entropy vanishes, $S = 0$. (Third law of thermodynamics.)

At zero temperature the energy can be written

$$E = \sum_{\alpha \text{ in Fermi sea}} \epsilon_\alpha = 2 \sum_{\mathbf{k},|\mathbf{k}|<k_F} \frac{\hbar^2 k^2}{2m}.$$

Where we used $\alpha = (\mathbf{k}, \sigma)$ and $\epsilon_\alpha = \epsilon_{|\mathbf{k}|} = \frac{\hbar^2 k^2}{2m}$. In the limit $L \to \infty$ the $\mathbf{k}$ sum can be transformed to an integral by noting that one $\mathbf{k}$-point occupies the volume $\Delta\mathbf{k} = (2\pi/L)^3$:

$$\sum_{\mathbf{k}} = (\frac{L}{2\pi})^3 \sum_{\mathbf{k}} \Delta\mathbf{k} \to (\frac{L}{2\pi})^3 \int d^3k.$$

Now since the energies only depend on $|\mathbf{k}|$, the angular integral just gives 4π and we have

$$E = 2(\frac{L}{2\pi})^3 4\pi \int_0^{k_F} k^2 dk \frac{\hbar^2 k^2}{2m} = \frac{L^3}{\pi^2}\frac{\hbar^2}{2m} \int_0^{k_F} k^4 dk = \frac{\hbar^2}{2m}\frac{L^3}{\pi^2}\frac{k_F^5}{5}$$

Similarly

$$N = \sum_{\alpha \text{ in Fermi sea}} = 2 \sum_{\mathbf{k},|\mathbf{k}|<k_F} = \frac{L^3}{\pi^2} \int_0^{k_F} k^2 dk = \frac{L^3}{\pi^2}\frac{k_F^3}{3}$$

Dividing these and using $\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$ we find

$$\frac{E}{N} = \frac{3}{5}\epsilon_F.$$

5. (DEMO – no need to calculate) In the lecture notes we identified the Fermi distribution from the expression for the total particle number for noninteracting fermions. Show, using a similar calculation, that the Fermi and Bose distributions follow also directly from

$$\langle \hat{n}_\alpha \rangle = \mathrm{Tr}[\hat{n}_\alpha e^{\beta(\Omega - \hat{H} + \mu \hat{N})}] = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} \pm 1}$$

Here $\hat{H} = \sum_\alpha \epsilon_\alpha \hat{n}_\alpha$ and $\hat{N} = \sum_\alpha \hat{n}_\alpha$, where $\hat{n}_\alpha$ is the number operator for the single-particle level α and ϵ_α is its energy. For fermions (upper sign) $\hat{n}_\alpha$ has the eigenvalues $n_\alpha = 0, 1$ and for bosons (lower sign) $n_\alpha = 0, 1, 2, \ldots$.

Solution:

In the lecture notes it was shown that for fermions

$$e^{-\beta\Omega} = \mathrm{Tr}[e^{-\beta(\hat{H} - \mu\hat{N})}] = \prod_\alpha [1 + e^{-\beta(\epsilon_\alpha - \mu)}]$$

Now very similarly to that calculation, we have

$$\begin{aligned}
\langle \hat{n}_\alpha \rangle &= \mathrm{Tr}[\hat{n}_\alpha e^{\beta(\Omega - \hat{H} + \mu\hat{N})}] = e^{\beta\Omega}\mathrm{Tr}[\hat{n}_\alpha e^{-\beta(\hat{H} - \mu\hat{N})}] \\
&= e^{\beta\Omega} \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \cdots \langle n_1, n_2, \ldots | \hat{n}_\alpha e^{-\beta(\hat{H} - \mu\hat{N})}] | n_1, n_2, \ldots \rangle \\
&= e^{\beta\Omega} (\sum_{n_\alpha=0}^{1} n_\alpha e^{-\beta(\epsilon_\alpha - \mu) n_\alpha}) \prod_{\gamma \neq \alpha} \sum_{n_\gamma=0}^{1} e^{-\beta(\epsilon_\gamma - \mu) n_\gamma} \\
&= e^{\beta\Omega} e^{-\beta(\epsilon_\alpha - \mu)} \prod_{\gamma \neq \alpha} [1 + e^{-\beta(\epsilon_\gamma - \mu)}]
\end{aligned}$$

Then, inserting $e^{-\beta\Omega}$ from above

$$\langle \hat{n}_\alpha \rangle = \frac{e^{-\beta(\epsilon_\alpha - \mu)} \prod_{\gamma \neq \alpha} [1 + e^{-\beta(\epsilon_\gamma - \mu)}]}{\prod_\gamma [1 + e^{-\beta(\epsilon_\gamma - \mu)}]} = \frac{e^{-\beta(\epsilon_\alpha - \mu)}}{1 + e^{-\beta(\epsilon_\alpha - \mu)}} = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1}$$

The boson calculation is similar, giving first

$$e^{-\beta\Omega} = \mathrm{Tr}[e^{-\beta(\hat{H} - \mu\hat{N})}] = \prod_\alpha \frac{1}{1 - e^{-\beta(\epsilon_\alpha - \mu)}}$$

Then

$$\begin{aligned}
\langle \hat{n}_\alpha \rangle &= e^{\beta\Omega} \sum_{n_\alpha=0}^{\infty} n_\alpha e^{-\beta(\epsilon_1-\mu)n_\alpha} \prod_{\gamma\neq\alpha} \sum_{n_\gamma=0}^{\infty} [e^{-\beta(\epsilon_\gamma-\mu)}]^{n_\gamma} \\
&= -e^{\beta\Omega} [\frac{\partial}{\partial\beta(\epsilon_\alpha-\mu)} \sum_{n_\alpha=0}^{\infty} e^{-\beta(\epsilon_\alpha-\mu)n_\alpha}] \prod_{\gamma\neq\alpha} \sum_{n_\gamma=0}^{\infty} [e^{-\beta(\epsilon_\gamma-\mu)}]^{n_\gamma} \\
&= -e^{\beta\Omega} [\frac{\partial}{\partial\beta(\epsilon_\alpha-\mu)} \frac{1}{1-e^{-\beta(\epsilon_\alpha-\mu)}}] \prod_{\gamma\neq\alpha} \frac{1}{1-e^{-\beta(\epsilon_\gamma-\mu)}} \\
&= e^{\beta\Omega} \frac{e^{-\beta(\epsilon_\alpha-\mu)}}{(1-e^{\beta(\epsilon_\alpha-\mu)})^2} \prod_{\gamma\neq\alpha} \frac{1}{1-e^{-\beta(\epsilon_\gamma-\mu)}} = \frac{1}{e^{\beta(\epsilon_\alpha-\mu)}-1}
\end{aligned}$$