1. Derive the equation

$$-\,\mathrm{d}t \oint \mathrm{d}\mathbf{a}\cdot(\mathbf{E}\times\mathbf{H}) = \int dV(\mathbf{E}\cdot\,\mathrm{d}\mathbf{D}\,+\mathbf{H}\cdot\,\mathrm{d}\mathbf{B}\,+\mathbf{E}\cdot\mathbf{j}_F\,\mathrm{d}t\,).$$

Solution:
Here we are going to need the Gauss law (GL), a vector identity (VI) and two of the Maxwell equations (MEs):

$$\oint_S \mathrm{d}\mathbf{a}\cdot\mathbf{B} = \int_V \mathrm{d}V\,\boldsymbol{\nabla}\cdot\mathbf{B} \qquad \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$$
$$\boldsymbol{\nabla}\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} + \mathbf{j}_\mathrm{f} \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}.$$

Using these the derivation goes like

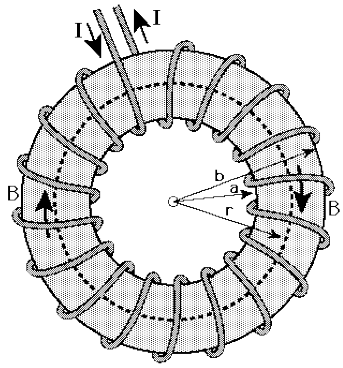
$$\begin{aligned}
-\,\mathrm{d}t \oint_S \mathrm{d}\mathbf{a}\cdot(\mathbf{E}\times\mathbf{H}) &\overbrace{=}^{\text{GL}} -\,\mathrm{d}t\int_V \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) \\
&\overbrace{=}^{\text{VI}} -\,\mathrm{d}t\int_V \mathrm{d}V\,[\mathbf{H}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) - \mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H})] \\
&\overbrace{=}^{\text{MEs}} -\,\mathrm{d}t\int_V \left[-\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\left(\frac{\partial\mathbf{D}}{\partial t} + \mathbf{j}_\mathrm{f}\right)\right] \\
&= \int_V \mathrm{d}V\,\left(\mathbf{H}\cdot\,\mathrm{d}\mathbf{B}\,+\mathbf{E}\cdot\,\mathrm{d}\mathbf{D}\,+\mathbf{E}\cdot\mathbf{j}_\mathrm{f}\,\mathrm{d}t\,\right)
\end{aligned}$$

Note: Here we have simply defined $\mathrm{d}\mathbf{B} = \frac{\partial\mathbf{B}}{\partial t}\,\mathrm{d}t$ etc. A “variational differential” notation $\delta\mathbf{B}$ might be more appropriate, since the differentials depend on position.

Note 2: This at least “justifies” the appearance of the “$V\mathbf{H}\cdot d\mathbf{B}$” term in the differential for internal energy (dE) and thus the Helmholtz free energy (dF), when we assume that $\frac{\partial\mathbf{D}}{\partial t}$ and the “free currents” $\mathbf{j}_f$ vanish. Existence of screening supercurrents induced by the applied magnetic field are still allowed, because they are not of the “free” variety. They are (I think) to be understood as the “magnetization currents” or “bound currents” $\nabla\times\mathbf{M}$ in Eq. (314) of the lecture notes.

2. In the lectures it was claimed that when the magnetic flux density $\mathbf{B}$ inside the sample changes, the current source has to do a work $V\mathbf{H}\cdot\,\mathrm{d}\mathbf{B}$ in order to keep the current going to the coil constant. Prove this more accurately. In order for the field to be homogeneous

inside the sample (with $b - a \ll r$) and zero elsewhere, it is easiest to think of a toroidal coil surrounding a toroidal sample (see figure).



Solution
The power of a current source is $P = UI$, where U is the voltage and I the current. The work done in a small time increment $\mathrm{d}t$ is $dW = P\,\mathrm{d}t = UI\,\mathrm{d}t$. This work should be now calculated for the toroidal coil sketched in Fig. Let us first calculate the current I and after that the voltage U.

Let us assume that there is a uniform $\mathbf{H}$ field inside the coil. Applying Maxwell equations with the ignorance of $\partial\mathbf{D}/\partial t$ we get

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{H} &= \mathbf{j}_\mathrm{f} \\ \int_S \mathrm{d}\mathbf{a} \cdot \boldsymbol{\nabla} \times \mathbf{H} &= \int_S \mathrm{d}\mathbf{a} \cdot \mathbf{j}_\mathrm{f} \\ \int_l \mathrm{d}\mathbf{l} \cdot \mathbf{H} &= NI \\ 2\pi r H &= NI \\ I &= \frac{2\pi r H}{N} \end{aligned}$$

where the path l is the boundary of the surface S and the surface S a circle of radius r inside the torus. N is the number of loops and I the current.

Let us now assume that the $\mathbf{B}$ field inside the coil changes although the current I is constant. This is possible if the material of the coil changes somehow, for example the temperature alters the permeability μ (which is what happens in the Meissner effect: μ drops to zero). The voltage U between the two ends of the wire going around the coil is path integral (following the wire) of the electric field:

$$U = -\int_r \mathrm{d}\mathbf{r} \cdot \mathbf{E} \underbrace{\approx}^{\text{Stokes theorem}} -N\int_A \mathrm{d}\mathbf{a} \cdot \nabla \times \mathbf{E}$$

$$\underbrace{=}^{ME} N\int_A \mathrm{d}\mathbf{a} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

$$= N\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathrm{d}\mathbf{a} \cdot \mathbf{B}$$

$$= N\frac{\mathrm{d}(BA)}{\mathrm{d}t}$$

where B is the magnitude of the uniform B field, A the cross-sectional area of the coil ($A = \pi(b-a)^2/4$, assuming a circular cross-section), and N the number of wire loops.

Now using $U\,\mathrm{d}t = NA\,\mathrm{d}B$ and $I = 2\pi rH/N$, the work is written explicitly:

$$dW = UI\,\mathrm{d}t = 2\pi rHA\,\mathrm{d}B = VH\,\mathrm{d}B\,.$$

with $V = 2\pi rA$. Furthermore, the fields $\mathbf{H}$ and $\mathbf{B}$ are parallel in a toroidal coil. In general we may expect

$$dW = V\mathbf{H}\cdot\,\mathrm{d}\mathbf{B}\,.$$

3. Calculate the latent heat and the change in the specific heat in a normal metal-superconductor phase transition, using

$$H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right].$$

Sketch the results as a function of temperature.

Solution
The latent heat is defined as the finite heat release/absorption occurring at a phase transition, during which the temperature remains constant. For the superconducting phase transition $\Delta Q = T\Delta S = T[S_s(T,H) - S_n(T,H)]$, where T is the considered transition temperature ($T = T_c$ at $H = 0$). In the lectures the change of entropy is considered and an expression for it is derived

$$\Delta S = S_s(T,H) - S_n(T,H) = V\mu_0 H_c(T)\frac{\mathrm{d}H_c(T)}{\mathrm{d}T}.$$

By using the expression given for the critical field, the latent heat is

$$\Delta Q = -2V\mu_0 H_c^2(0)\left[\left(\frac{T}{T_c}\right)^2 - \left(\frac{T}{T_c}\right)^4\right].$$

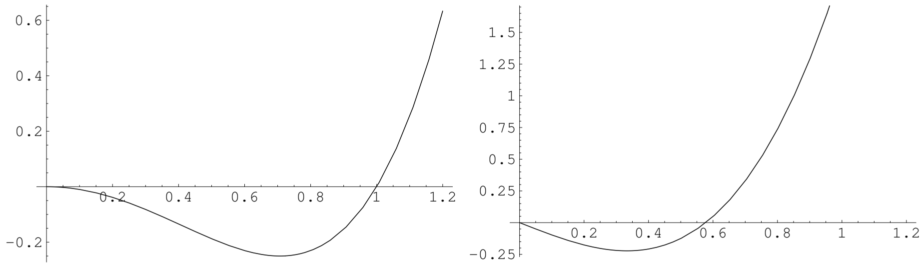


Figure 1: (LEFT) The latent heat (1) and (RIGHT) the specific heat (2). At critical temperature T_c where external H field must vanish, the superconducting phase transition is of the second order: vanishing latent heat ($\Delta Q = T\Delta S = 0$) but non-zero change in the specific heat ($\Delta C \neq 0$). In the region $0 < T < T_c$ the jump ΔQ is finite, and the phase transision is thus of first order. Interestingly, at $T \approx 0.58T_c$ the jump ΔC vanishes, but this is also a first-order case because ΔQ is finite.

This can be modified to a simpler form

$$\Delta Q = -A(x^2 - x^4). \tag{1}$$

where x is the reduced temperature $x = T/T_c$ and A includes all the other parameters.

Similarly for the discontinuity of the specific heat is derived an expression in lectures:

$$\Delta C = TV\mu_0 \left[\left(\frac{\mathrm{d}H_c(T)}{\mathrm{d}T} \right)^2 + H_c(T) \frac{\mathrm{d}^2 H_c(T)}{\mathrm{d}T^2} \right].$$

After calculating open the dervatives one arrives at the formula

$$\Delta C = \frac{2V\mu_0 H_c^2(0)}{T_c} \left(\frac{3T^3}{T_c^3} - \frac{T}{T_c} \right)$$

which is again simplified to the form

$$\Delta C = B(3x^3 - x). \tag{2}$$

The latent heat and specific heat are sketched in Fig. 1.

4. Consider a superconducting wire with a radius R. Calculate the maximum supercurrent I_c that can flow so that the field caused by the current itself does not exceed the critical field H_c at the surface of the wire.

Solution
From the Maxwell equation

$$\boldsymbol{\nabla} \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

one obtains, by using the Stokes theorem and the stationarity assumption $\partial \mathbf{E}/\partial t = 0$, that

$$\oint \mathbf{B} \cdot \mathrm{d}l = \mu_0 \int \mathrm{d}\mathbf{a} \cdot \mathbf{j}.$$

Integrating around a circle of radius r around the wire and assuming $\mathbf{B} = B(r)\hat{\boldsymbol{\theta}}$, we obtain

$$2\pi r B(r) = \mu_0 I,$$

where $I = \int \mathrm{d}\mathbf{a} \cdot \mathbf{j}$ is the total current flowing in the wire. Thus the field at $r > R$ is $B(r) = \mu_0 I/(2\pi r)$. (We need not care about what it is for $r < R$.)

The critical current I_c is the current which gives an $\mathbf{H}$ field of the critical magnitude H_c on the surface of the wire at $r = R$. Since outside the sample we have simply $\mathbf{B} = \mu_0\mathbf{H}$, this corresponds to a $\mathbf{B}$ field of magnitude $B_c = \mu_0 H_c$. So the critical current satisfies $2\pi R B_c = \mu_0 I_c$, giving

$$I_c = 2\pi R B_c/\mu_0 = 2\pi R H_c. \tag{3}$$

5. Estimate the Fermi temperature for aluminum using the mass density $\rho = 2.7$ g/cm^3, the atomic weight and assuming 3 (noninteracting) conducting electrons/atom with an effective mass of a free electron m_e. Calculate the ratio T_c/T_F.

Solution

In lectures it was stated a relation between Fermi wave vector k_F and electron density $\rho = N/V$:

$$k_F = (3\pi^2\rho)^{1/3}.$$

The Fermi temperature is defined through the Fermi energy:

$$T_F = \frac{\epsilon_F}{k_B} = \frac{\hbar^2 k_F^2}{2m_e k_B} = \frac{\hbar^2(3\pi^2\rho)^{2/3}}{2m_e k_B}$$

The atomic mass of Al is $27u$, with $u = 1.6605 \cdot 10^{-27}$ kg, and the mass density was given as 2700 kg/m^3. Thus the number density of aluminun electron gas is

$$\rho = 3\rho_{at} = 3\frac{2700 \text{ kg/m}^3}{27u} = 1.806 \cdot 10^{29} \text{ 1/m}^3.$$

Now $m_e = 9.10938 \cdot 10^{-31}$ kg, $\hbar = 1.054571 \cdot 10^{-34}$ Js, and $k_B = 1.38065 \cdot 10^{-23}$ J/K. After calculations we get $T_F = 1.36 \cdot 10^5$ K. The critical temperature for Al is $T_c = 1.196$ K. The ratio

$$\frac{T_c}{T_F} = \frac{1.196 \text{ K}}{1.36 \cdot 10^5 \text{ K}} = 9 \cdot 10^{-6}$$

then tells us that the electron system is very degenerate at the temperatures relevant for superconductivity: essentially only states below the Fermi surface, $k < k_F$, are occupied.