1. Show that for an arbitrary function $g(\mathbf{k})$

$$\frac{1}{L^3} \sum_{\mathbf{k}} g(\mathbf{k}) = \int \frac{d^2\Omega}{4\pi} \int d\epsilon N(\epsilon) g(\mathbf{k}),$$

and argue that if $g(\mathbf{k}) \neq 0$ only near the Fermi surface, one obtains the equation shown in the lectures.

Solution:
First some preliminaries. The number of states with wave vectors of length less than k (hence the subscript $<$) is $N_<(k) = \frac{4\pi}{3}k^3/(\frac{2\pi}{L})^3$. Here $\frac{4\pi}{3}k^3$ is the volume of a sphere of radius k and $(\frac{2\pi}{L})^3$ is the volume occupied by one allowed $\mathbf{k}$ point. The number $\mathrm{d}N_<(k)$ of states between spherical surfaces at radii k and $k + \mathrm{d}k$ is just

$$\mathrm{d}N_<(k) = N'_<(k)\,\mathrm{d}k = 4\pi k^2 (\frac{L}{2\pi})^3\,\mathrm{d}k = \frac{L^3 k^2\,\mathrm{d}k}{2\pi^2}.$$

The relation between energy ϵ and wave vector k, $\epsilon = \hbar^2 k^2/(2m)$, may be applied to put these quantities in terms of energy. Thus, using

$$k = (\frac{2m\epsilon}{\hbar^2})^{1/2}, \qquad\qquad \mathrm{d}k = \frac{\sqrt{m}}{\hbar\sqrt{2\epsilon}}\,\mathrm{d}\epsilon,$$

we have $N_<(\epsilon) = \frac{4\pi}{3}(\frac{2m\epsilon}{\hbar^2})^{3/2}(\frac{L}{2\pi})^3$ and

$$\mathrm{d}N_<(\epsilon) = N'_<(\epsilon)d\epsilon = L^3 \frac{m^{3/2}\sqrt{\epsilon}}{\hbar^3\sqrt{2}\pi^2}\,\mathrm{d}\epsilon = L^3 N(\epsilon)\,\mathrm{d}\epsilon,$$

where $N(\epsilon)$ is the energy density of states at energy ϵ per unit volume of the system: $N(\epsilon) = \frac{1}{L^3}\frac{\mathrm{d}N_<(\epsilon)}{d\epsilon}$. Note that $\mathrm{d}N_<(k) = N'_<(k)\,\mathrm{d}k = N'_<(\epsilon)\,\mathrm{d}\epsilon = \mathrm{d}N_<(\epsilon) = L^3 N(\epsilon)\,\mathrm{d}\epsilon$.

Now for the actual transformation. Let us separate the $\mathbf{k}$ dependence in $g(\mathbf{k})$ to a dependence on $k = |\mathbf{k}|$ and on $\hat{\mathbf{k}} = \mathbf{k}/k$, and write $g(\mathbf{k}) = g(\hat{\mathbf{k}}, k)$. Here k can be written as a function of ϵ. So, replacing the $\mathbf{k}$ sum by an integral and remembering that for spherical coordinates the volume element is $\mathrm{d}^3 k = \mathrm{d}\Omega_{\hat{\mathbf{k}}} k^2\,\mathrm{d}k$, where $\mathrm{d}\Omega_{\hat{\mathbf{k}}} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ (θ and ϕ being the polar and azimuthal angles of $\hat{\mathbf{k}}$), we develop

$$\frac{1}{L^3}\sum_{\mathbf{k}} g(\mathbf{k}) = \int \frac{d^3 k}{(2\pi)^3} g(\mathbf{k}) = \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \int \frac{k^2 dk}{2\pi^2} g(\hat{\mathbf{k}}, k)$$

$$= \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \int \frac{\mathrm{d}N_<(k)}{L^3} g(\hat{\mathbf{k}}, k) = \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \int \frac{\mathrm{d}N_<(\epsilon)}{L^3} g(\hat{\mathbf{k}}, k(\epsilon))$$

$$= \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \int N(\epsilon) g(\hat{\mathbf{k}}, k(\epsilon))\,\mathrm{d}\epsilon = \int \frac{d^2\Omega_{\hat{\mathbf{k}}}}{4\pi} \int N(\epsilon) g(\mathbf{k})\,\mathrm{d}\epsilon.$$

Here the k and ϵ integrals are all from 0 to ∞. The last form is the desired one — note that a dependence of $\mathbf{k}$ on ϵ, θ, and ϕ is implied.

If $g(\mathbf{k}) \neq 0$ "only near $k = k_F$", then the density of states $N(\epsilon)$ may be approximated by its Fermi-surface value $N(\epsilon_F) = mk_F/(\hbar^2 2\pi^2)$ in the whole region where the integrand is finite. The constant factor $N(\epsilon_F)$ can then be taken out of the integral such that

$$\frac{1}{L^3}\sum_{\mathbf{k}} g(\mathbf{k}) = N(\epsilon_F) \int \frac{d^2\Omega}{4\pi} \int g(\mathbf{k})\, d\epsilon\,.$$

Choosing the zero of energy at ϵ_F in all of the above definitions, like $k^2 = 2m(\epsilon + \epsilon_F)/\hbar^2$, you can replace $N(\epsilon_F) \to N(0)$.

Note: There is another way of deriving the result that may be useful to mention. Let's assume that we can write $g(\mathbf{k}) = g(\hat{\mathbf{k}}, \epsilon_k)$. Then

$$\frac{1}{L^3}\sum_{\mathbf{k}} g(\mathbf{k}) = \int \frac{d^3k}{(2\pi)^3} g(\hat{\mathbf{k}}, \epsilon_k) = \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} g(\hat{\mathbf{k}}, \epsilon_k)$$

$$= \int d\epsilon \underbrace{\int \frac{k^2 dk}{2\pi^2}\delta(\epsilon - \epsilon_k)}_{N(\epsilon)} \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} g(\hat{\mathbf{k}}, \epsilon) = \int d\epsilon N(\epsilon) \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} g(\hat{\mathbf{k}}, \epsilon),$$

because the (single-spin) density of states is always given by $N(\epsilon) = \frac{1}{L^3}\sum_{\mathbf{k}} \delta(\epsilon - \epsilon_{\mathbf{k}}) = \int \frac{d^3k}{(2\pi)^3}\delta(\epsilon - \epsilon_{\mathbf{k}})$ regardless of what the dispersion $\epsilon_{\mathbf{k}}$ is.

Note 2: The approximation to the general result can be made more precise by expanding $N(\epsilon) = N(\epsilon_F) + N'(\epsilon_F)(\epsilon - \epsilon_F) + O[(\epsilon - \epsilon_F)^2]$ and deducing that the neglected terms are of higher order in the small parameter $T/T_F \ll 1$, assuming that the energy width of g is given by temperature T.

2. Particles with integer and half-integer total spin follow boson and fermion statistics, respectively. By calculating the numbers of protons, neutrons and electrons (which all have spin $1/2$), deduce which of the following particles are bosons and which of them are fermions:
$$^1\text{H},\ ^3\text{He},\ ^4\text{He},\ ^6\text{Li},\ ^7\text{Li},\ ^{23}\text{Na},\ ^{87}\text{Rb}.$$

Solution:
The table lists the isotopes $^A X$ of element X, with atomic mass number A, the number of protons N_p (usually called the atomic number Z), the number of neutrons N_n (usually denoted N), and the number of electrons N_e:

Atom	N_p	N_n	N_e	fermion/boson nature
^{1}H	1	0	1	boson
^{3}He	2	1	2	fermion
^{4}He	2	2	2	boson
^{6}Li	3	3	3	fermion
^{7}Li	3	4	3	boson
^{23}Na	11	12	11	boson
^{87}Rb	37	50	37	boson

These satisfy $A = N_p + N_n$ and $N_e = N_p$. Protons, neutrons, and electrons all carry a spin $1/2$. The atom is a boson if the total spin is an integer, otherwise it is a fermion. An even number of spin $1/2$s always yields a total spin that is an integer and an odd number leads to a half-integer. (See rules for quantum-mechanical addition of angular momenta if needed.) Now, the total number of the spin-carrying particles is $N_n + N_p + N_e = N_n + 2N_p$. Since $2N_p$ is even, only the number of neutrons N_n really counts.

Let us take ^{1}H as an example. It consists of two spin-$1/2$ particles. The total spin of two spin-$1/2$ particles is either 0 or 1, so in any case the total spin is integer and ^{1}H atom has bosonic nature in many particle statistics. The second example is ^{3}He-atom which consists of five spin-$1/2$ particles. Five halves cannot be summed to any integer number. The ^{3}He-atom is thus considered as fermion.

3. In the course of statistical physics it is shown, that the Bose condensation (for spin=0 particles) occurs below the temperature

$$T_{\text{BOSE}} = 3.31 \frac{\hbar^2}{m k_B} \left(\frac{N}{V} \right)^{2/3},$$

where m is the mass of the boson and N/V the number density. Using the mass densities of ^{3}He and ^{4}He ($\rho_3 = 0.081$ g/cm^3 and $\rho_4 = 0.145$ g/cm^3) calculate the Fermi temperature for ^{3}He and the Bose condensation temperature for ^{4}He, and compare them with the superfluid transition temperatures $T_3 = 0.93$ mK and $T_4 = 2.17$ K observed for the two helium isotopes.

Solution:
In the previous problem the ^{3}He atom was deduced as fermion and ^{4}He as boson. Thus for ^{3}He the Fermi temperature is calculated and for ^{4}He the Bose condensation temperature. Given values for densities and atomic masses:

$$\rho_3 = 81 \text{ kg/}m^3, \qquad \rho_4 = 145 \text{ kg/}m^3, \qquad m_3 = 3u, \qquad m_4 = 4u$$

where $u = 1.66053 \cdot 10^{-27}$ kg is the atomic mass unit. The number density is then naturally $N/V = \rho/m$.

The Bose condensation temperature $T_{\text{BOSE},4}$ for ^{4}He (using $\hbar = 1.054571 \cdot 10^{-34}$ Js and $k_B = 1.38065 \cdot 10^{-23}$ J/K):

$$T_{\text{BOSE},4} = 3.31 \frac{\hbar^2}{m_4 k_B} \left(\frac{\rho_4}{m_4}\right)^{2/3} = 3.31 \frac{\hbar^2}{m_4^{5/3} k_B} \rho_4^{2/3} = 3.1 \text{ K}$$

which is of the same order with the experimental transition temperature for superfluidity, $T_4 = 2.17 K$.

The Fermi temperature T_F for ^{3}He is calculated from the chain of definitions: $\epsilon_F = k_B T_F$, $\epsilon_F = \hbar^2 k_F^2/(2m_3)$ and $k_F = (3\pi^2 N/V)^{1/3}$. So it reads:

$$T_F = \frac{\hbar^2}{2m_3^{5/3} k_B}(3\pi^2 \rho_3)^{2/3} = 4.9 \text{ K}.$$

The superfluid transition temperature is $T_3 = 0.93$ mK. The ratio of these two is $T_3/T_F = 1.9 \cdot 10^{-4}$. This is similar to T_c/T_F in the case of superconducting transition of electrons in a metal.

As a conclusion, superfluidity in ^{4}He probably has something to do with Bose-Einstein condensation and superfluidity in ^{3}He with Cooper pairing as in superconductivity. The case of ^{3}He may be discussed later in the lectures.

4. Check all the intermediate steps in the Cooper problem shown in the lecture notes.

(Hint: Orthonormality of plane waves $\int e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}} d^3 r = L^3 \delta_{\mathbf{k},\mathbf{q}}$.)

Solution:
Let us start with the Schrödinger equation (SE) [Eq. (85) in the notes]

$$\left[-\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + V(\mathbf{r}_1, \mathbf{r}_2)\right] \phi(\mathbf{r}_1, \mathbf{r}_2) = E\phi(\mathbf{r}_1, \mathbf{r}_2) \tag{1}$$

As mentioned in the lecture, we assume the center of mass of the pair to be at rest and thus write the wave function as

$$\phi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{L^3}\sum_{\mathbf{k}} \chi(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}. \tag{2}$$

After plugging this into SE and calculating the derivatives $(\nabla_1^2 + \nabla_2^2)$ we are left with

$$\frac{1}{L^3}\sum_{\mathbf{k}} \chi(\mathbf{k}) \left[2 \overbrace{\frac{\hbar^2 k^2}{2m}}^{\epsilon_k} + V(\mathbf{r}_1, \mathbf{r}_2)\right] e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} = \frac{E}{L^3}\sum_{\mathbf{k}} \chi(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \tag{3}$$

which after a little manipulation reads

$$\frac{1}{L^3}\sum_{\mathbf{k}} (2\epsilon_k - E)\,\chi(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} = -\frac{1}{L^3}\sum_{\mathbf{k}} \chi(\mathbf{k}) V(\mathbf{r}_1, \mathbf{r}_2) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}. \tag{4}$$

Now the above equation is multiplied with $L^3 e^{-i\mathbf{q}\cdot(\mathbf{r}_1-\mathbf{r}_2)}$ and after that integrated over both spaces $\mathbf{r}_1$ and $\mathbf{r}_2$

$$\sum_{\mathbf{k}} (2\epsilon_k - E)\, \chi(\mathbf{k}) \underbrace{\int\int e^{-i(\mathbf{q}-\mathbf{k})\cdot\mathbf{r}_1 + i(\mathbf{q}-\mathbf{k})\cdot\mathbf{r}_2}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2}_{(L^3\delta_{\mathbf{q},\mathbf{k}})^2 = L^6\delta_{\mathbf{q},\mathbf{k}}}$$

$$= -\sum_{\mathbf{k}} \chi(\mathbf{k}) \underbrace{\int\int V(\mathbf{r}_1,\mathbf{r}_2) e^{-i(\mathbf{q}-\mathbf{k})\cdot\mathbf{r}_1 + i(\mathbf{q}-\mathbf{k})\cdot\mathbf{r}_2}\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2}_{L^6\langle \mathbf{q},-\mathbf{q}|V|\mathbf{k},-\mathbf{k}\rangle}\, . \tag{5}$$

where we used the notation in Eq. (88) of the lecture notes. The first identification with a Kronecker delta arises from elementary complex integration and can be seen representing the Fourier transform of unity, or orthonormality of plane waves under the assumption of periodic boundary conditions: $\int d^3 r\, e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}} = L^3\delta_{\mathbf{k},\mathbf{q}}$ (prove this!). Thus we have [Eq. (87) of the notes]

$$(2\epsilon_q - E)\, \chi(\mathbf{q}) = -\sum_{\mathbf{k}} \langle \mathbf{q},-\mathbf{q}|V|\mathbf{k},-\mathbf{k}\rangle \chi(\mathbf{k}) \tag{6}$$

Here we should insert [(90) of notes]

$$\langle \mathbf{q},-\mathbf{q}|V|\mathbf{k},-\mathbf{k}\rangle = \begin{cases} -\frac{g}{L^3} & |\epsilon_k - \epsilon_F| < \epsilon_c, |\epsilon_q - \epsilon_F| < \epsilon_c \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Then (6) reduces to

$$(2\epsilon_q - E)\, \chi(\mathbf{q}) = I \tag{8}$$

where the right-hand is just a constant (independent of q as long as $|\epsilon_q - \epsilon_F| < \epsilon_c$):

$$I = \frac{g}{L^3} \sum_{\mathbf{k},\epsilon_F - \epsilon_c < \epsilon_k < \epsilon_F + \epsilon_c} \chi(\mathbf{k}) \tag{9}$$

From (8) we deduce [(91) of notes]

$$\chi(\mathbf{q}) = \frac{I}{2\epsilon_q - E}\theta(q - k_F), \tag{10}$$

where the Heaviside step function $\theta(x)$ is now added to take into account the Pauli principle: $\chi(\mathbf{q}) = 0$ for $q < k_F$. (See note below.) Inserting (10) into (9) and noting that Is cancel we have [(92) of notes]

$$1 = \frac{g}{L^3} \sum_{\mathbf{k},\epsilon_F < \epsilon_k < \epsilon_F + \epsilon_c} \frac{1}{2\epsilon_k - E} \tag{11}$$

which yields, upon applying Eq. (76) of the notes,

$$\frac{1}{N(0)g} = \int_{\epsilon_F}^{\epsilon_F + \epsilon_c} d\epsilon \frac{1}{2\epsilon - E} \tag{12}$$

This gives [(93) of notes]

$$\frac{1}{N(0)g} = \frac{1}{2} \ln \frac{2\epsilon_F - E + 2\epsilon_c}{2\epsilon_F - E} \tag{13}$$

Now if $2\epsilon_F - E \ll 2\epsilon_c$, then the ratio reduces just to form

$$\frac{2\epsilon_F - E + 2\epsilon_c}{2\epsilon_F - E} = \frac{2\epsilon_c}{2\epsilon_F - E}.$$

With this reduction the equation (13) is easy to solve and see that the energy E given in the equation [(94) of notes]:

$$E = 2\epsilon_F - 2\epsilon_c e^{-2/gN(0)} \tag{14}$$

Note: The requirement that $\chi(\mathbf{k}) = 0$ for $k < k_F$ is an *ad hoc* procedure that needs to be introduced at some point in the calculation, because we are trying to describe a many-body problem as a two-body one. We could just as well have restricted to $k, q > k_F$ from the beginning, replacing also $|\epsilon_{k,q} - \epsilon_F| < \epsilon_c$ in (7) with $0 < \epsilon_{k,q} - \epsilon_F < \epsilon_c$. (See de Gennes.) That's not what is done in the notes, because in the general BCS many-body calculation to be considered later on, the interaction should be finite also for $k < k_F$.

Note 2: One of the main results of the Cooper calculation is that on the Fermi surface the pairs can form with an arbitrarily weak attractive interaction. This should be contrasted with the case of two isolated particles, which is basically obtained from the Cooper problem by shrinking the Fermi sphere to a point: $k_F = \epsilon_F = 0$. Then the density of states is $N(\epsilon) \propto \sqrt{\epsilon}$ and it cannot be approximated with a constant around "$\epsilon = \epsilon_F$" (that is $\epsilon = 0$). In this case it is (apparently) found that the interaction must exceed some threshold in order for pairs to form. See de Gennes.

Note 3: The original reference for this calculation is L. N. Cooper, Phys. Rev. **104**, 1189–1190 (1956). The full BCS theory followed a year later, in J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175–1204 (1957).

5. Think of Cooper's problem for a repulsive interaction, i.e. $g < 0$ in the notation of the lecture notes. Where does the calculation differ from the case of an attractive interaction, $g > 0$? In both cases you can look at the weak-coupling limit $|g|N(0) \ll 1$.

   (Hint: You can actually solve for the energy of the pair state without assumptions on the sign or magnitude of $gN(0)$. After that it's easier to consider the special cases.)

   Solution: The energy E can be solved from Eq. (93) of the lecture notes without any approximations. Using a shorthand notation $A = 2/gN(0)$:

$$\frac{1}{gN(0)} = \frac{1}{2} \ln \left(\frac{2\epsilon_F - E + 2\epsilon_c}{2\epsilon_F - E} \right)$$

$$e^A = \left(\frac{2\epsilon_F - E + 2\epsilon_c}{2\epsilon_F - E} \right)$$

$$E = 2\epsilon_F - \frac{2}{e^A - 1} \epsilon_c$$

Attractive interaction means $g > 0$. Then, in the weak-coupling limit $gN(0) \ll 1 \rightarrow A \gg 1$, which gives $e^A \gg 1$ and therefore

$$E_{\text{att}} = 2\epsilon_F - \frac{2}{e^A - 1}\epsilon_c \approx 2\epsilon_F - 2e^{-A}\epsilon_c$$

as solved in lectures. When the interaction is repulsive, $g < 0$. Now the weak-coupling limit is $|g|N(0) \ll 1 \rightarrow A \ll -1$, leading to $e^A \approx 0$ and thus

$$E_{\text{rep}} = 2\epsilon_F - \frac{2}{e^A - 1}\epsilon_c \approx 2\epsilon_f + 2\epsilon_c$$

which is larger than $2\epsilon_F$ and thus the pair 'formed' by repulsive interaction is not stable.