1. **Fermion Hamiltonian in plane wave basis**
The matrix elements including spin variables are defined

$$\langle \boldsymbol{k}_1\lambda_1|T|\boldsymbol{k}_2\lambda_2\rangle = \sum_\sigma \int d^3r\, \phi^*_{\boldsymbol{k}_1\lambda_1}(\boldsymbol{r},\sigma)\left(-\frac{\hbar^2}{2m}\nabla^2\right)\phi_{\boldsymbol{k}_2\lambda_2}(\boldsymbol{r},\sigma),$$

$$\langle \boldsymbol{k}_1\lambda_1\boldsymbol{k}_2\lambda_2|V|\boldsymbol{k}_3\lambda_3\boldsymbol{k}_4\lambda_4\rangle$$
$$= \sum_\sigma\sum_{\sigma'}\int d^3r\int d^3r'\, \phi^*_{\boldsymbol{k}_1\lambda_1}(\boldsymbol{r},\sigma)\phi^*_{\boldsymbol{k}_2\lambda_2}(\boldsymbol{r}',\sigma')V(\boldsymbol{r}-\boldsymbol{r}')\phi_{\boldsymbol{k}_3\lambda_3}(\boldsymbol{r},\sigma)\phi_{\boldsymbol{k}_4\lambda_4}(\boldsymbol{r}',\sigma').$$

[Here we have assumed a spin-independent translation-invariant potential $V(\boldsymbol{r},\boldsymbol{r}') = V(\boldsymbol{r}-\boldsymbol{r}')$.] Calculate the matrix elements using plane waves

$$\phi_{\boldsymbol{k}\lambda}(\boldsymbol{r},\sigma) = \frac{1}{L^{3/2}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\delta_{\lambda\sigma}. \tag{1}$$

Show that the general Hamiltonian given in the lecture notes reduces to

$$\check{H} = \sum_{\boldsymbol{k},\sigma}\epsilon_k\check{a}^\dagger_{\boldsymbol{k}\sigma}\check{a}_{\boldsymbol{k}\sigma} + \frac{1}{2L^3}\sum_{\boldsymbol{k}_1,\sigma}\sum_{\boldsymbol{k}_2,\lambda}\sum_{\boldsymbol{k}_3}\sum_{\boldsymbol{k}_4}V(\boldsymbol{k}_1-\boldsymbol{k}_3)\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4}\check{a}^\dagger_{\boldsymbol{k}_1\sigma}\check{a}^\dagger_{\boldsymbol{k}_2\lambda}\check{a}_{\boldsymbol{k}_4\lambda}\check{a}_{\boldsymbol{k}_3\sigma}.$$

Here σ, $\lambda = \uparrow$ or $\downarrow$ and $V(\boldsymbol{k}) = \int d^3rV(\boldsymbol{r})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$.

(Hint: Change to coordinates $\tilde{\boldsymbol{r}} = \boldsymbol{r}-\boldsymbol{r}'$ and $\boldsymbol{R} = (\boldsymbol{r}+\boldsymbol{r}')/2$ in the double integral: $\int d^3r\int d^3r' = \int d^3R\int d^3\tilde{r}$.)

Solution:

In plane wave basis

$$\phi_{\boldsymbol{k}\lambda}(\boldsymbol{r},\sigma) = \frac{1}{L^{3/2}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\delta_{\lambda\sigma}. \tag{2}$$

The matrix element of the usual kinetic energy operator $T = -\hbar^2\nabla^2/2m$:

$$
\begin{aligned}
\langle \boldsymbol{k}_1\lambda_1|T|\boldsymbol{k}_2\lambda_2\rangle &= \sum_\sigma \int d^3r\, \phi^*_{\boldsymbol{k}_1\lambda_1}(\boldsymbol{r},\sigma)\left(-\frac{\hbar^2}{2m}\nabla^2\right)\phi_{\boldsymbol{k}_2\lambda_2}(\boldsymbol{r},\sigma) \\
&= \sum_\sigma \int d^3r\, \frac{1}{L^{3/2}}e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}}\delta_{\lambda_1\sigma}\left(-\frac{\hbar^2}{2m}\nabla^2\right)\frac{1}{L^{3/2}}e^{i\boldsymbol{k}_2\cdot\boldsymbol{r}}\delta_{\lambda_2\sigma} \\
&= \left(\sum_\sigma \delta_{\lambda_1\sigma}\delta_{\lambda_2\sigma}\right)\left(-\frac{\hbar^2}{2m}\right)\frac{1}{L^3}\int d^3r e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}}\nabla^2 e^{i\boldsymbol{k}_2\cdot\boldsymbol{r}} \\
&= \delta_{\lambda_1\lambda_2}\left(-\frac{\hbar^2}{2m}\right)\frac{1}{L^3}\int d^3r e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}}\left(-k_2^2\right)e^{i\boldsymbol{k}_2\cdot\boldsymbol{r}} \\
&= \delta_{\lambda_1\lambda_2}\frac{\hbar^2k_2^2}{2m}\underbrace{\frac{1}{L^3}\int d^3r e^{i(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}}}_{=\delta_{\boldsymbol{k}_1\boldsymbol{k}_2}} \\
&= \delta_{\lambda_1\lambda_2}\left(-\frac{\hbar^2}{2m}\right)\frac{1}{L^3}\int d^3r e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}}\left(-k_2^2\right)e^{i\boldsymbol{k}_2\cdot\boldsymbol{r}} \\
&= \delta_{\lambda_1\lambda_2}\delta_{\boldsymbol{k}_1\boldsymbol{k}_2}\frac{\hbar^2k_2^2}{2m}.
\end{aligned}
$$

Thus the kinetic energy part of the Hamiltonian $\check{H} = \check{T} + \check{V}$ is

$$
\begin{aligned}
\check{T} &= \sum_{\boldsymbol{k}_1\lambda_1}\sum_{\boldsymbol{k}_2\lambda_2}\langle \boldsymbol{k}_1\lambda_1|T|\boldsymbol{k}_2\lambda_2\rangle \check{a}^\dagger_{\boldsymbol{k}_1\lambda_1}\check{a}_{\boldsymbol{k}_2\lambda_2} = \sum_{\boldsymbol{k}_1\lambda_1}\sum_{\boldsymbol{k}_2\lambda_2}\delta_{\lambda_1\lambda_2}\delta_{\boldsymbol{k}_1\boldsymbol{k}_2}\frac{\hbar^2k_2^2}{2m}\check{a}^\dagger_{\boldsymbol{k}_1\lambda_1}\check{a}_{\boldsymbol{k}_2\lambda_2} \\
&= \sum_{\boldsymbol{k}_1\lambda_1}\frac{\hbar^2k_1^2}{2m}\check{a}^\dagger_{\boldsymbol{k}_1\lambda_1}\check{a}_{\boldsymbol{k}_1\lambda_1} = \sum_{\boldsymbol{k}\sigma}\frac{\hbar^2k^2}{2m}\check{a}^\dagger_{\boldsymbol{k}\sigma}\check{a}_{\boldsymbol{k}\sigma} = \sum_{\boldsymbol{k}\sigma}\epsilon_k\check{a}^\dagger_{\boldsymbol{k}\sigma}\check{a}_{\boldsymbol{k}\sigma}
\end{aligned}
$$

Let us move on to calculate the potential energy part. Since the potential is spin-independent and translation-invariant, $V(\boldsymbol{r},\boldsymbol{r}') = V(\boldsymbol{r}-\boldsymbol{r}')$, we have

$$
\begin{aligned}
&\langle \boldsymbol{k}_1\lambda_1\boldsymbol{k}_2\lambda_2|V|\boldsymbol{k}_3\lambda_3\boldsymbol{k}_4\lambda_4\rangle \\
&= \sum_\sigma\sum_{\sigma'}\int d^3r\int d^3r'\, \phi^*_{\boldsymbol{k}_1\lambda_1}(\boldsymbol{r},\sigma)\phi^*_{\boldsymbol{k}_2\lambda_2}(\boldsymbol{r}',\sigma')V(\boldsymbol{r}-\boldsymbol{r}')\phi_{\boldsymbol{k}_3\lambda_3}(\boldsymbol{r},\sigma)\phi_{\boldsymbol{k}_4\lambda_4}(\boldsymbol{r}',\sigma') \\
&= \sum_\sigma\sum_{\sigma'}\int d^3r\int d^3r'\, \frac{1}{L^{3/2}}e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}}\delta_{\lambda_1\sigma}\frac{1}{L^{3/2}}e^{-i\boldsymbol{k}_2\cdot\boldsymbol{r}'}\delta_{\lambda_2\sigma'}V(\boldsymbol{r}-\boldsymbol{r}')\frac{1}{L^{3/2}}e^{i\boldsymbol{k}_3\cdot\boldsymbol{r}}\delta_{\lambda_3\sigma}\frac{1}{L^{3/2}}e^{i\boldsymbol{k}_4\cdot\boldsymbol{r}'}\delta_{\lambda_4\sigma'} \\
&= \sum_\sigma\delta_{\lambda_1\sigma}\delta_{\lambda_3\sigma}\sum_{\sigma'}\delta_{\lambda_2\sigma'}\delta_{\lambda_4\sigma'}\frac{1}{L^6}\int d^3r\int d^3r'\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}}e^{-i\boldsymbol{k}_2\cdot\boldsymbol{r}'}V(\boldsymbol{r}-\boldsymbol{r}')e^{i\boldsymbol{k}_3\cdot\boldsymbol{r}}e^{i\boldsymbol{k}_4\cdot\boldsymbol{r}'} \\
&= \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\frac{1}{L^6}\int d^3r\int d^3r'\, e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_3)\cdot\boldsymbol{r}}e^{-i(\boldsymbol{k}_2-\boldsymbol{k}_4)\cdot\boldsymbol{r}'}V(\boldsymbol{r}-\boldsymbol{r}')
\end{aligned} \tag{3}
$$

The integral can be simplified by transforming to center-of-mass and relative coordinates $\mathbf{R}, \widetilde{\mathbf{r}}$:

$$\mathbf{r} = \mathbf{R} + \frac{1}{2}\widetilde{\mathbf{r}} \qquad \mathbf{r}' = \mathbf{R} - \frac{1}{2}\widetilde{\mathbf{r}}.$$

The integral then reads (*Note*: the Jacobian determinant equals 1)

$$\underbrace{\int \mathrm{d}^3\mathbf{R}\, \mathrm{e}^{-\mathrm{i}(\mathbf{k_1}+\mathbf{k_2}-(\mathbf{k_3}+\mathbf{k_4}))\cdot\mathbf{R}}}_{L^3\delta_{\mathbf{k_1}+\mathbf{k_2},\mathbf{k_3}+\mathbf{k_4}}} \int \mathrm{d}^3\widetilde{\mathbf{r}}\, \mathrm{e}^{-\mathrm{i}(\mathbf{k_1}-\mathbf{k_3})\cdot\frac{1}{2}\widetilde{\mathbf{r}}} V(\widetilde{\mathbf{r}}) \mathrm{e}^{\mathrm{i}(\mathbf{k_2}-\mathbf{k_4})\cdot\frac{1}{2}\widetilde{\mathbf{r}}}$$

Now the first delta function says that $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \Rightarrow \mathbf{k}_1 - \mathbf{k}_3 = -(\mathbf{k}_2 - \mathbf{k}_4)$. Thus the whole integral has form

$$L^3\delta_{\mathbf{k_1}+\mathbf{k_2},\mathbf{k_3}+\mathbf{k_4}} \int \mathrm{d}^3\widetilde{\mathbf{r}}\, \mathrm{e}^{-\mathrm{i}(\mathbf{k_1}-\mathbf{k_3})\cdot\widetilde{\mathbf{r}}} V(\widetilde{\mathbf{r}}) = L^3\delta_{\mathbf{k_1}+\mathbf{k_2},\mathbf{k_3}+\mathbf{k_4}} V(\mathbf{k}_1 - \mathbf{k}_3).$$

Hence the matrix element is

$$\begin{aligned}
&\langle \boldsymbol{k}_1\lambda_1\boldsymbol{k}_2\lambda_2|V|\boldsymbol{k}_3\lambda_3\boldsymbol{k}_4\lambda_4\rangle \\
&= \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\frac{1}{L^6}\int d^3r \int d^3r'\, e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_3)\cdot\boldsymbol{r}} e^{-i(\boldsymbol{k}_2-\boldsymbol{k}_4)\cdot\boldsymbol{r}'} V(\boldsymbol{r}-\boldsymbol{r}') \\
&= \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\frac{1}{L^6} L^3 \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4} V(\boldsymbol{k}_1 - \boldsymbol{k}_3) \\
&= \frac{1}{L^3}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4} V(\boldsymbol{k}_1 - \boldsymbol{k}_3)
\end{aligned} \tag{4}$$

Using this matrix element we get the interaction part of $\check{H} = \check{T} + \check{V}$:

$$\begin{aligned}
\check{V} =& \frac{1}{2}\sum_{\boldsymbol{k}_1\lambda_1}\sum_{\boldsymbol{k}_2\lambda_2}\sum_{\boldsymbol{k}_3\lambda_3}\sum_{\boldsymbol{k}_4\lambda_4} \langle \boldsymbol{k}_1\lambda_1\boldsymbol{k}_2\lambda_2|V|\boldsymbol{k}_3\lambda_3\boldsymbol{k}_4\lambda_4\rangle \check{a}^\dagger_{\boldsymbol{k}_1\lambda_1}\check{a}^\dagger_{\boldsymbol{k}_2\lambda_2} a_{\boldsymbol{k}_4\lambda_4} a_{\boldsymbol{k}_3\lambda_3} \\
=& \frac{1}{2}\sum_{\boldsymbol{k}_1\lambda_1}\sum_{\boldsymbol{k}_2\lambda_2}\sum_{\boldsymbol{k}_3\lambda_3}\sum_{\boldsymbol{k}_4\lambda_4} \frac{1}{L^3}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4} V(\boldsymbol{k}_1 - \boldsymbol{k}_3)\check{a}^\dagger_{\boldsymbol{k}_1\lambda_1}\check{a}^\dagger_{\boldsymbol{k}_2\lambda_2} a_{\boldsymbol{k}_4\lambda_4} a_{\boldsymbol{k}_3\lambda_3} \\
=& \frac{1}{2L^3}\sum_{\boldsymbol{k}_1\lambda_1}\sum_{\boldsymbol{k}_2\lambda_2}\sum_{\boldsymbol{k}_3}\sum_{\boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4} V(\boldsymbol{k}_1 - \boldsymbol{k}_3)\check{a}^\dagger_{\boldsymbol{k}_1\lambda_1}\check{a}^\dagger_{\boldsymbol{k}_2\lambda_2} a_{\boldsymbol{k}_4\lambda_2} a_{\boldsymbol{k}_3\lambda_1}.
\end{aligned}$$

We change the summation indices as $\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_3$, $\boldsymbol{k}_2 \leftrightarrow \boldsymbol{k}_4$, $\lambda_1 \rightarrow \sigma$ and $\lambda_2 \rightarrow \lambda$, and thus obtain

$$\check{V} = \frac{1}{2L^3}\sum_{\boldsymbol{k}_1\sigma}\sum_{\boldsymbol{k}_2\lambda}\sum_{\boldsymbol{k}_3}\sum_{\boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4} V(\boldsymbol{k}_3 - \boldsymbol{k}_1)\check{a}^\dagger_{\boldsymbol{k}_3\sigma}\check{a}^\dagger_{\boldsymbol{k}_4\lambda} a_{\boldsymbol{k}_2\lambda} a_{\boldsymbol{k}_1\sigma}.$$

Hence the Hamiltonian $\check{H} = \check{T} + \check{V}$ is

$$\check{H} = \sum_{\boldsymbol{k}\sigma}\epsilon_k \check{a}^\dagger_{\boldsymbol{k}\sigma}\check{a}_{\boldsymbol{k}\sigma} + \frac{1}{2L^3}\sum_{\boldsymbol{k}_1\sigma}\sum_{\boldsymbol{k}_2\lambda}\sum_{\boldsymbol{k}_3}\sum_{\boldsymbol{k}_4} V(\boldsymbol{k}_3 - \boldsymbol{k}_1)\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4}\check{a}^\dagger_{\boldsymbol{k}_3\sigma}\check{a}^\dagger_{\boldsymbol{k}_4\lambda} a_{\boldsymbol{k}_2\lambda} a_{\boldsymbol{k}_1\sigma}.$$

2. **Contact interaction**
As a special case of the previous exercise, consider a contact interaction $V(\boldsymbol{r}) = -g\delta(\boldsymbol{r})$. Write the Hamiltonian in this case. Show that terms with $\sigma = \lambda$ vanish, and those with $\sigma \neq \lambda$ are equal, as mentioned in the lecture notes.

Solution:
In the case $V(\mathbf{r}) = -g\delta(\mathbf{r})$ we clearly have the Fourier transform $V(\mathbf{k}) = -g$. Thus

$$\check{V} = \frac{-g}{2L^3} \sum_{\mathbf{k}_1,\sigma} \sum_{\mathbf{k}_2,\lambda} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4} \check{a}^\dagger_{\mathbf{k}_3,\sigma} \check{a}^\dagger_{\mathbf{k}_4,\lambda} \check{a}_{\mathbf{k}_2,\lambda} \check{a}_{\mathbf{k}_1,\sigma}.$$

Then if you look at one of the two terms with $\lambda = \sigma$ and use the anticommutation rule $\check{a}_{\mathbf{k}_2,\sigma}\check{a}_{\mathbf{k}_1,\sigma} = -\check{a}_{\mathbf{k}_1,\sigma}\check{a}_{\mathbf{k}_2,\sigma}$, and finally relabel $\mathbf{k}_1 \leftrightarrow \mathbf{k}_2$, you get the same term but with a negative sign. For example for the term $\lambda = \sigma = \uparrow$

$$\begin{aligned}
\text{term}(\lambda = \sigma = \uparrow) &= \frac{-g}{2L^3} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4} \check{a}^\dagger_{\mathbf{k}_3,\uparrow} \check{a}^\dagger_{\mathbf{k}_4,\uparrow} \check{a}_{\mathbf{k}_2,\uparrow} \check{a}_{\mathbf{k}_1,\uparrow} \\
&= -\frac{-g}{2L^3} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4} \check{a}^\dagger_{\mathbf{k}_3,\uparrow} \check{a}^\dagger_{\mathbf{k}_4,\uparrow} \check{a}_{\mathbf{k}_1,\uparrow} \check{a}_{\mathbf{k}_2,\uparrow} \\
&= -\frac{-g}{2L^3} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} \delta_{\mathbf{k}_2+\mathbf{k}_1,\mathbf{k}_3+\mathbf{k}_4} \check{a}^\dagger_{\mathbf{k}_3,\uparrow} \check{a}^\dagger_{\mathbf{k}_4,\uparrow} \check{a}_{\mathbf{k}_2,\uparrow} \check{a}_{\mathbf{k}_1,\uparrow} \\
&= -\text{term}(\lambda = \sigma = \uparrow).
\end{aligned}$$

Thus the terms with $\lambda = \sigma$ must be zero, and you are left with

$$\check{V} = \frac{-g}{2L^3} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4} (\check{a}^\dagger_{\mathbf{k}_3,\uparrow} \check{a}^\dagger_{\mathbf{k}_4,\downarrow} \check{a}_{\mathbf{k}_2,\downarrow} \check{a}_{\mathbf{k}_1,\uparrow} + \check{a}^\dagger_{\mathbf{k}_3,\downarrow} \check{a}^\dagger_{\mathbf{k}_4,\uparrow} \check{a}_{\mathbf{k}_2,\uparrow} \check{a}_{\mathbf{k}_1,\downarrow}).$$

Again here in the second term we may anticommute the operators so that $\check{a}^\dagger_{\mathbf{k}_3,\downarrow} \check{a}^\dagger_{\mathbf{k}_4,\uparrow} \check{a}_{\mathbf{k}_2,\uparrow} \check{a}_{\mathbf{k}_1,\downarrow} = \check{a}^\dagger_{\mathbf{k}_4,\uparrow} \check{a}^\dagger_{\mathbf{k}_3,\downarrow} \check{a}_{\mathbf{k}_1,\downarrow} \check{a}_{\mathbf{k}_2,\uparrow}$ and then by relabeling $\mathbf{k}_1 \leftrightarrow \mathbf{k}_2$ $\mathbf{k}_3 \leftrightarrow \mathbf{k}_4$ we see that the two terms are equal. Thus

$$\check{V} = -\frac{g}{L^3} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4} \check{a}^\dagger_{\mathbf{k}_3,\uparrow} \check{a}^\dagger_{\mathbf{k}_4,\downarrow} \check{a}_{\mathbf{k}_2,\downarrow} \check{a}_{\mathbf{k}_1,\uparrow}.$$

The Hamiltonian is

$$\check{H} = \sum_{\boldsymbol{k}\sigma} \epsilon_k \check{a}^\dagger_{\boldsymbol{k}\sigma} \check{a}_{\boldsymbol{k}\sigma} - \frac{g}{L^3} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4} \check{a}^\dagger_{\mathbf{k}_3,\uparrow} \check{a}^\dagger_{\mathbf{k}_4,\downarrow} \check{a}_{\mathbf{k}_2,\downarrow} \check{a}_{\mathbf{k}_1,\uparrow}.$$

3. **Bogoliubov transformation**
Do all the intermediate steps of the Bogoliubov transformation [Eqs. (148)-(156)] not shown in the lecture notes. That is, check carefully the result $\check{K}_{\text{eff}} = \sum_{\boldsymbol{k}\sigma} E_k \check{\gamma}^\dagger_{\boldsymbol{k}\sigma} \check{\gamma}_{\boldsymbol{k}\sigma} + \Omega_0$.

What is Ω_0? You can save yourself some work by not following precisely the route implied in the lecture notes, but rather starting from the matrix form (147), and inserting

$$\begin{pmatrix} \check{a}_{\boldsymbol{k}\uparrow} \\ \check{a}^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix} = U_k \begin{pmatrix} \check{\gamma}_{\boldsymbol{k}\uparrow} \\ \check{\gamma}^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u_k^* & v_k \\ -v_k^* & u_k \end{pmatrix} \begin{pmatrix} \check{\gamma}_{\boldsymbol{k}\uparrow} \\ \check{\gamma}^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix}.$$

Require the coefficient matrix U_k to be unitary ($U_k U_k^\dagger = U_k^\dagger U_k = 1$), and then require that $U_k^\dagger K_k U_k$ is diagonal, where K_k is the Hermitian matrix appearing in the Hamiltonian. These two requirements give you the equations for u_k and v_k [(150) and (152)]. When solving them, you can assume Δ to be real, as in the lecture notes.

Solution 1:
The Bogoliubov transformation is defined to be

$$\check{a}_{\mathbf{k}\uparrow} = u_k^* \check{\gamma}_{\mathbf{k}\uparrow} + v_k \check{\gamma}^\dagger_{-\mathbf{k}\downarrow} \qquad \check{a}_{\mathbf{k}\downarrow} = u_k^* \check{\gamma}_{\mathbf{k}\downarrow} - v_k \check{\gamma}^\dagger_{-\mathbf{k}\uparrow} \tag{148}$$

$$\check{a}^\dagger_{\mathbf{k}\uparrow} = u_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow} + v_k^\star \check{\gamma}_{-\mathbf{k}\downarrow} \qquad \check{a}^\dagger_{\mathbf{k}\downarrow} = u_k \check{\gamma}^\dagger_{\mathbf{k}\downarrow} - v_k^\star \check{\gamma}_{-\mathbf{k}\uparrow}. \tag{149}$$

Let us first show the inverse transformation (151) by substituting terms from (148)-(149) to the inverse transformation:

$$\begin{aligned} \check{\gamma}_{\mathbf{k}\uparrow\downarrow} &= u_k \check{a}_{\mathbf{k}\uparrow\downarrow} \mp v_k \check{a}^\dagger_{\mathbf{k}\downarrow\uparrow} = u_k(u_k^* \check{\gamma}_{\mathbf{k}\uparrow\downarrow} \pm v_k \check{\gamma}^\dagger_{-\mathbf{k}\downarrow\uparrow}) \mp v_k(u_k \check{\gamma}^\dagger_{-\mathbf{k}\downarrow\uparrow} \mp v_k^\star \check{\gamma}_{\mathbf{k}\uparrow\downarrow}) \\ &= \underbrace{(|u_k|^2 + |v_k|^2)}_{=1,\text{ see Eq. (150)}} \check{\gamma}_{\mathbf{k}\uparrow\downarrow} + u_k v_k \check{\gamma}^\dagger_{-\mathbf{k}\downarrow\uparrow} - u_k v_k \check{\gamma}^\dagger_{-\mathbf{k}\downarrow\uparrow} = \check{\gamma}_{\mathbf{k}\uparrow\downarrow}. \end{aligned}$$

To express the spin dependence of $\check{\gamma}$ operators a shorthand notation is applied

$$\check{\gamma}_{\mathbf{k}\alpha} = u_k \check{a}_{\mathbf{k}\alpha} - \alpha v_k \check{a}^\dagger_{-\mathbf{k}-\alpha} \qquad \check{\gamma}^\dagger_{\mathbf{k}\alpha} = u_k^* \check{a}^\dagger_{\mathbf{k}\alpha} - \alpha v_k^\star \check{a}_{-\mathbf{k}-\alpha}$$

where α can have values $\pm$ which naturally correspond values $\uparrow\downarrow$. Using this notation the anticommutators are calculated in quite a compact notation:

$$\begin{aligned} \{\check{\gamma}_{\mathbf{k}\alpha}, \check{\gamma}^\dagger_{\mathbf{k}'\beta}\} =& \{u_k \check{a}_{\mathbf{k}\alpha} - \alpha v_k \check{a}^\dagger_{-\mathbf{k}-\alpha}, u_{k'}^* \check{a}^\dagger_{\mathbf{k}'\beta} - \beta v_{k'}^\star \check{a}_{-\mathbf{k}'-\beta}\} \\ =& u_k u_{k'}^* \underbrace{\{\check{a}_{\mathbf{k}\alpha}, \check{a}^\dagger_{\mathbf{k}'\beta}\}}_{=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\beta}} + \alpha\beta v_k v_{k'}^\star \underbrace{\{\check{a}^\dagger_{-\mathbf{k}-\alpha}, \check{a}_{-\mathbf{k}'-\beta}\}}_{=\delta_{-\mathbf{k},-\mathbf{k}'}\delta_{-\alpha,-\beta}} \\ & - \alpha v_k u_{k'}^* \underbrace{\{\check{a}^\dagger_{-\mathbf{k}-\alpha}, \check{a}^\dagger_{\mathbf{k}'\beta}\}}_{=0} - \beta v_{k'}^\star u_k \underbrace{\{\check{a}_{\mathbf{k}\alpha}, \check{a}_{-\mathbf{k}'-\beta}\}}_{=0} \\ =& \delta_{\mathbf{k},\mathbf{k}'}(|u_k|^2 \delta_{\alpha,\beta} + \alpha\beta\delta_{\alpha,\beta}|v_k|^2) = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\beta} \underbrace{(|u_k|^2 + |v_k|^2)}_{=1} \\ =& \delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\beta} \end{aligned}$$

and

$$\begin{aligned}
\{\check{\gamma}_{\mathbf{k}\alpha}, \check{\gamma}_{\mathbf{k}'\beta}\} =& \{u_k \check{a}_{\mathbf{k}\alpha} - \alpha v_k \check{a}^\dagger_{-\mathbf{k}-\alpha}, u_{k'} \check{a}_{\mathbf{k}'\beta} - \beta v_{k'} \check{a}^\dagger_{-\mathbf{k}'-\beta}\} \\
=& u_k u_{k'} \underbrace{\{\check{a}_{\mathbf{k}\alpha}, \check{a}_{\mathbf{k}'\beta}\}}_{=0} + \alpha\beta v_k v_{k'} \underbrace{\{\check{a}^\dagger_{-\mathbf{k}-\alpha}, \check{a}^\dagger_{-\mathbf{k}'-\beta}\}}_{=0} \\
& - \alpha v_k u_{k'} \underbrace{\{\check{a}^\dagger_{-\mathbf{k}-\alpha}, \check{a}_{\mathbf{k}'\beta}\}}_{=\delta_{-\mathbf{k},\mathbf{k}'}\delta_{-\alpha,\beta}} - \beta v_{k'} u_k \underbrace{\{\check{a}_{\mathbf{k}\alpha}, \check{a}^\dagger_{-\mathbf{k}'-\beta}\}}_{=\delta_{\mathbf{k},-\mathbf{k}'}\delta_{\alpha,-\beta}} \\
=& \delta_{-\mathbf{k},\mathbf{k}'}\delta_{-\alpha,\beta}(\beta v_k u_k - \beta v_k u_k) \\
=& 0
\end{aligned}$$

In the calculation of the anticommutator $\{\check{\gamma}_{\mathbf{k}\alpha}, \check{\gamma}_{\mathbf{k}'\beta}\}$, the fact that the coefficients u and v are dependent on only the length of the vector $\mathbf{k}$ is applied: $v_{|\mathbf{k}|} = v_{|-\mathbf{k}|} = v_k$ etc. The anticommutator $\{\check{\gamma}^\dagger_{\mathbf{k}\alpha}, \check{\gamma}^\dagger_{\mathbf{k}'\beta}\}$ is a hermitian conjugate of the anticommutator $\{\check{\gamma}_{\mathbf{k}\alpha}, \check{\gamma}_{\mathbf{k}'\beta}\}$.

The next task is a brave and brute substitution of Bogoliubov transformation to the Hamiltonian $\check{K}_{\text{eff}}$ of Eq. (144). Our goal is to express $\check{K}_{\text{eff}}$ in terms of $\check{\gamma}$ instead of $\check{a}$. The original Hamiltonian reads

$$\check{K}_{\text{eff}} = \sum_{\mathbf{k}} \xi_k (\check{a}^\dagger_{\mathbf{k}\uparrow}\check{a}_{\mathbf{k}\uparrow} + \check{a}^\dagger_{\mathbf{k}\downarrow}\check{a}_{\mathbf{k}\downarrow}) - \sum_{\mathbf{k}} \left(\Delta \check{a}^\dagger_{\mathbf{k}\uparrow}\check{a}^\dagger_{-\mathbf{k}\downarrow} + \Delta^\star \check{a}_{-\mathbf{k}\downarrow}\check{a}_{\mathbf{k}\uparrow}\right) + C$$

Here the constant term is $C = \frac{L^3}{g}|\Delta|^2$. For simplicity we shall drop it in the following, but it should be kept in mind for later! In the calculation we will need four types of expressions

$$\begin{aligned}
\check{a}^\dagger_{\mathbf{k}\uparrow}\check{a}_{\mathbf{k}\uparrow} &= (u_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow} + v^\star_k \check{\gamma}_{-\mathbf{k}\downarrow})(u^*_k \check{\gamma}_{\mathbf{k}\uparrow} + v_k \check{\gamma}^\dagger_{-\mathbf{k}\downarrow}) = |u_k|^2 \check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} + |v_k|^2 \check{\gamma}_{-\mathbf{k}\downarrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow} + u_k v_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow} + u^*_k v^\star_k \check{\gamma}_{-\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\uparrow} \\
\check{a}^\dagger_{\mathbf{k}\downarrow}\check{a}_{\mathbf{k}\downarrow} &= (u_k \check{\gamma}^\dagger_{\mathbf{k}\downarrow} - v^\star_k \check{\gamma}_{-\mathbf{k}\uparrow})(u^*_k \check{\gamma}_{\mathbf{k}\downarrow} - v_k \check{\gamma}^\dagger_{-\mathbf{k}\uparrow}) = |u_k|^2 \check{\gamma}^\dagger_{\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\downarrow} + |v_k|^2 \check{\gamma}_{-\mathbf{k}\uparrow}\check{\gamma}^\dagger_{-\mathbf{k}\uparrow} - u_k v_k \check{\gamma}^\dagger_{\mathbf{k}\downarrow}\check{\gamma}^\dagger_{-\mathbf{k}\uparrow} - u^*_k v^\star_k \check{\gamma}_{-\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\downarrow} \\
\check{a}^\dagger_{\mathbf{k}\uparrow}\check{a}^\dagger_{-\mathbf{k}\downarrow} &= (u_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow} + v^\star_k \check{\gamma}_{-\mathbf{k}\downarrow})(u_k \check{\gamma}^\dagger_{-\mathbf{k}\downarrow} - v^\star_k \check{\gamma}_{\mathbf{k}\uparrow}) = u^2_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow} - u_k v^\star_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} + v^\star_k u_k \check{\gamma}_{-\mathbf{k}\downarrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow} - (v^\star_k)^2 \check{\gamma}_{-\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\uparrow} \\
\check{a}_{-\mathbf{k}\downarrow}\check{a}_{\mathbf{k}\uparrow} &= (\check{a}^\dagger_{\mathbf{k}\uparrow}\check{a}^\dagger_{-\mathbf{k}\downarrow})^\dagger = (u^*_k)^2 \check{\gamma}_{-\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\uparrow} - u^*_k v_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} + v_k u^*_k \check{\gamma}_{-\mathbf{k}\downarrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow} - v^2_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow}
\end{aligned}$$

The next step is to substitute the above expressions to $\check{K}_{\text{eff}}$. We can apply two simplifying tricks: 1) Coefficients are dependent only on length of $\mathbf{k}$ 2) The anticommutator rules just derived for $\check{\gamma}$ are available. So, a couple of manipulations in fashion of

$$\sum_{\mathbf{k}} \xi_k |v_k|^2 \check{\gamma}_{-\mathbf{k}\downarrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow} = \sum_{\mathbf{k}} \xi_k |v_k|^2 \check{\gamma}_{\mathbf{k}\downarrow}\check{\gamma}^\dagger_{\mathbf{k}\downarrow} = \sum_{\mathbf{k}} \xi_k (|v_k|^2 - |v_k|^2 \check{\gamma}^\dagger_{\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\downarrow})$$

are made. The end result of the substitution and manipulation is

$$\begin{aligned}
\check{K}_{\rm eff} =& \sum_{\mathbf{k}} \xi_k \left[(|u_k|^2 - |v_k|^2)(\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} + \check{\gamma}^\dagger_{\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\downarrow}) + 2|v_k|^2 + 2u_k v_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow} + 2u^*_k v^\star_k \check{\gamma}_{\mathbf{k}\downarrow}\check{\gamma}_{-\mathbf{k}\uparrow}\right] \\
&- \sum_{\mathbf{k}} \left\{-[\Delta^\star u^*_k v_k + \Delta u_k v^\star_k](\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} - \check{\gamma}_{\mathbf{k}\downarrow}\check{\gamma}^\dagger_{\mathbf{k}\downarrow}) + (\Delta u^2_k - \Delta^\star v^2_k)\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow} + [\Delta^\star (u^*_k)^2 - \Delta(v^\star_k)^2]\check{\gamma}_{\mathbf{k}\downarrow}\check{\gamma}_{-\mathbf{k}\uparrow}\right\} \\
=& \sum_{\mathbf{k}} \left[\xi_k(|u_k|^2 - |v_k|^2)(\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} + \check{\gamma}^\dagger_{\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\downarrow}) + (\Delta^\star u^*_k v_k + \Delta u_k v^\star_k)(\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} - \check{\gamma}_{\mathbf{k}\downarrow}\check{\gamma}^\dagger_{\mathbf{k}\downarrow})\right] \\
&+ \sum_{\mathbf{k}} (2\xi_k u_k v_k - \Delta u^2_k + \Delta^\star v^2_k)\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}^\dagger_{-\mathbf{k}\downarrow} + [2\xi_k u^*_k v^\star_k - \Delta^\star(u^*_k)^2 + \Delta(v^\star_k)^2]\check{\gamma}_{\mathbf{k}\uparrow}\check{\gamma}_{-\mathbf{k}\downarrow} \\
&+ \sum_{\mathbf{k}} 2\xi_k |v_k|^2.
\end{aligned}$$

Where the second row vanishes when Eq. (152) is required to hold, and the third row is just a constant.

We have to solve Eq. (152) $2\xi_k u_k v_k - \Delta u_k^2 + \Delta^\star v_k^2 = 0$ together with Eq. (150) $|u_k|^2 + |v_k|^2 = 1$. Assuming now that Δ (and hence u_k and v_k) are real, then

$$\begin{aligned}
2\xi_k u_k v_k - \Delta u^2_k + \Delta v^2_k &= 0 \\
2\xi_k \frac{\Delta v_k}{u_k} - \Delta^2 + \left(\frac{\Delta v_k}{u_k}\right)^2 &= 0 \\
2\xi_k x - \Delta^2 + x^2 &= 0 \\
x = -\xi_k \pm \sqrt{\xi_k^2 + \Delta^2} &= -\xi_k + E_k
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta v_k}{u_k} &= -\xi_k + E_k \\
\Delta^2 v^2_k &= (-\xi_k + E_k)^2(1 - v^2_k) \\
v^2_k &= \frac{(-\xi_k + E_k)^2}{\Delta^2 + (-\xi_k + E_k)^2} = \frac{(-\xi_k + E_k)^2}{2E_k(E_k - \xi_k)} = \frac{-\xi_k + E_k}{2E_k} = \frac{1}{2}(1 - \frac{\xi_k}{E_k})
\end{aligned}$$

and then from Eq. (150) $u^2_k = \frac{1}{2}(1 + \frac{\xi_k}{E_k})$. Above we chose the plus sign without comment. You can check that the minus sign exchanges the results for u_k^2 and v_k^2, which is unacceptable on physical grounds. (The Bogoliubov transformation should reduce to the normal-state canonical transformation, with $v_k^2 = 1$ for $k < k_F$.)

Finally, inserting the solutions obtained for u_k and v_k into the remaining nonvanishing terms of Hamiltonian we find

$$
\begin{aligned}
\check{K}_{\text{eff}} &= \sum_{\mathbf{k}} \Big[\xi_k(|u_k|^2 - |v_k|^2)(\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} + \check{\gamma}^\dagger_{\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\downarrow}) + (\Delta^\star u_k^* v_k + \Delta u_k v_k^\star)(\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} - \check{\gamma}_{\mathbf{k}\downarrow}\check{\gamma}^\dagger_{\mathbf{k}\downarrow})\Big] + \sum_{\mathbf{k}} 2\xi_k|v_k|^2 \\
&= \sum_{\mathbf{k}} \underbrace{[\underbrace{\xi_k(|u_k|^2 - |v_k|^2)}_{=\xi_k^2/E_k} + \underbrace{(\Delta^\star u_k^* v_k + \Delta u_k v_k^\star)}_{\Delta^2/E_k}]}_{=E_k}(\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} + \check{\gamma}^\dagger_{\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\downarrow}) + \underbrace{\sum_{\mathbf{k}}[2\xi_k|v_k|^2 - (\Delta^\star u_k^* v_k + \Delta u_k v^\star)]}_{=\tilde{C}} \\
&= \sum_{\mathbf{k}\sigma} E_k \check{\gamma}^\dagger_{\mathbf{k}\sigma}\check{\gamma}_{\mathbf{k}\sigma} + \tilde{C}
\end{aligned}
$$

Here the constant may be seen to be equal to $\tilde{C} = \sum_{\mathbf{k}}(\xi_k - E_k)$. We already neglected another such constant term earlier, which we called C. Adding that one to $\tilde{C}$, the total neglected constant is

$$
\begin{aligned}
\Omega_0 = C + \tilde{C} &= \sum_{\mathbf{k}}[2\xi_k|v_k|^2 - (\Delta^* u_k^* v_k + \Delta u_k v_k^*)] + \frac{g}{L^3}\sum_{\mathbf{k},\mathbf{k}'}\langle a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow}\rangle\langle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow}\rangle \\
&= \sum_{\mathbf{k}}[2\xi_k|v_k|^2 - (\Delta^* u_k^* v_k + \Delta u_k v_k^*)] + \frac{g}{L^3}\sum_{\mathbf{k},\mathbf{k}'} + \frac{L^3}{g}|\Delta|^2 \\
&= \sum_{\mathbf{k}}(\xi_k - E_k) + \frac{L^3}{g}|\Delta|^2
\end{aligned}
$$

This constant is not of relevance here, but in other contexts it is.

Solution 2:
The effective Hamiltonian can be written as

$$
K_{eff} = \sum_{\mathbf{k}} \begin{pmatrix} a^\dagger_{\mathbf{k}\uparrow} & a_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_k & -\Delta \\ -\Delta^* & -\xi_k \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}\uparrow} \\ a^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} + D
$$

where the constant term is now $D = \sum_{\mathbf{k}} \xi_{\mathbf{k}} + \frac{L^3}{g}|\Delta|^2$. By defining

$$
K_k = \begin{pmatrix} \xi_k & -\Delta \\ -\Delta^* & -\xi_k \end{pmatrix}, \quad \boldsymbol{a}_{\mathbf{k}} = \begin{pmatrix} a_{\mathbf{k}\uparrow} \\ a^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix}
$$

we have

$$
K_{eff} = \sum_{\mathbf{k}} \boldsymbol{a}^\dagger_{\mathbf{k}} K_k \boldsymbol{a}_{\mathbf{k}} + D
$$

The matrix K_k is hermitian and is diagonalizable by a unitary transformation. Thus we introduce the transformation (this follows directly from the equations in the notes)

$$
\begin{pmatrix} a_{\mathbf{k}\uparrow} \\ a^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u_k^* & v_k \\ -v_k^* & u_k \end{pmatrix} \begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix}
$$

or, in a compact notation, $\boldsymbol{a}_{\mathbf{k}} = U_k \boldsymbol{\gamma}_{\mathbf{k}}$. In this way

$$K_{eff} = \sum_{\mathbf{k}} \boldsymbol{\gamma}_{\mathbf{k}}^\dagger U_k^\dagger K_k U_k \boldsymbol{\gamma}_{\mathbf{k}} + D$$

Requiring U_k to be unitary, that is $U_k U_k^\dagger = U_k^\dagger U_k = 1$, is equivalent to requiring $|u_k^2| + |v_k|^2 = 1$. (You can check this. It can then also be checked that the transformation is not only unitary, but also canonical in the sense that it preserves the fermion commutation relations.) Note that since now $U_k^{-1} = U_k^\dagger$, the inverse transformation follows very easily: $\boldsymbol{\gamma}_{\mathbf{k}} = U_k^\dagger \boldsymbol{a}_{\mathbf{k}}$.

Now, applying the transformation on the matrix K_k yields

$$U_k^\dagger K_k U_k = \begin{pmatrix} \xi_k(|u_k|^2 - |v_k|^2) + (\Delta^* u_k^* v_k + \Delta u_k v_k^*) & 2\xi_k u_k v_k - \Delta u_k^2 + \Delta^* v_k^2 \\ 2\xi_k u_k^* v_k^* - \Delta^* (u_k^2)^* + \Delta (v_k^2)^* & -\xi_k(|u_k|^2 - |v_k|^2) - (\Delta^* u_k^* v_k + \Delta u_k v_k^*) \end{pmatrix}$$

Requiring this to be diagonal, i.e.

$$U_k^\dagger K_k U_k = \begin{pmatrix} d_k & 0 \\ 0 & -d_k \end{pmatrix}$$

where $d_k = \xi_k(|u_k|^2 - |v_k|^2) + (\Delta^* u_k^* v_k + \Delta u_k v_k^*)$ leads to the condition

$$2\xi_k u_k v_k - \Delta u_k^2 + \Delta^* v_k^2 = 0$$

Combining this with the unitarity condition $|u_k|^2 + |v_k|^2 = 1$ allows one to solve for u_k and v_k. (Here one may assume for simplicity that Δ is real.) The calculation is given above, and leads to the result $d_k = E_k = \sqrt{\xi_k^2 + |\Delta|^2}$. So now we have

$$K_{eff} = \sum_{\mathbf{k}} \begin{pmatrix} \gamma_{\mathbf{k}\uparrow}^\dagger & \gamma_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix} \begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix} + D$$

Upon application of the anticommutator and changing $\mathbf{k} \to -\mathbf{k}$ this gives

$$K_{eff} = \sum_{\mathbf{k},\sigma} E_k \gamma_{\mathbf{k}\sigma}^\dagger \gamma_{\mathbf{k}\sigma} + \Omega_0$$

where the constant is $\Omega_0 = D - \sum_{\mathbf{k}} E_k$, with $D = \sum_{\mathbf{k}} \xi_{\mathbf{k}} + \frac{L^3}{g}|\Delta|^2$, just as above. In the future it will be best to use the form $\Omega_0 = D - \sum_{\mathbf{k}} d_k$, where $d_k = \xi_k(|u_k|^2 - |v_k|^2) + (\Delta^* u_k^* v_k + \Delta u_k v_k^*)$, and where $|u_k|^2 = 1 - |v_k|^2$ etc. and v_k is kept free. Minimization of Ω_0 with respect to v_k then leads to the correct values for v_k and u_k and hence $d_k = E_k$.

Solution 3:

There is at least yet another way of proceeding, which is similar to the first case in spirit, but less troublesome. The goal is to insert the inverse transformation

$$\check{\gamma}_{\mathbf{k}\uparrow} = u_k \check{a}_{\mathbf{k}\uparrow} - v_k \check{a}_{-\mathbf{k}\downarrow}^\dagger, \qquad \check{\gamma}_{\mathbf{k}\uparrow}^\dagger = u_k^* \check{a}_{\mathbf{k}\uparrow}^\dagger - v_k^\star \check{a}_{-\mathbf{k}\downarrow}$$

$$\check{\gamma}_{\mathbf{k}\downarrow} = u_k \check{a}_{\mathbf{k}\downarrow} + v_k \check{a}_{-\mathbf{k}\uparrow}^\dagger, \qquad \check{\gamma}_{\mathbf{k}\downarrow}^\dagger = u_k^* \check{a}_{\mathbf{k}\downarrow}^\dagger + v_k^\star \check{a}_{-\mathbf{k}\uparrow}$$

into

$$K_{eff} = \sum_{\mathbf{k}} E_k(\check{\gamma}^\dagger_{\mathbf{k}\uparrow}\check{\gamma}_{\mathbf{k}\uparrow} + \check{\gamma}^\dagger_{\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\downarrow}) + \Omega_0$$

and identify the terms of the result with the terms of

$$\check{K}_{\text{eff}} = \sum_{\mathbf{k}} \xi_k(\check{a}^\dagger_{\mathbf{k}\uparrow}\check{a}_{\mathbf{k}\uparrow} + \check{a}^\dagger_{\mathbf{k}\downarrow}\check{a}_{\mathbf{k}\downarrow}) - \sum_{\mathbf{k}} \left(\Delta\check{a}^\dagger_{\mathbf{k}\uparrow}\check{a}^\dagger_{-\mathbf{k}\downarrow} + \Delta^\star\check{a}_{-\mathbf{k}\downarrow}\check{a}_{\mathbf{k}\uparrow}\right) + C$$

(You have to relabel $\mathbf{k} \leftrightarrow -\mathbf{k}$ in some terms to show that they are equal.) This gives now the following equations for u_k and v_k:

$$E_k(|u_k|^2 - |v_k|^2) = \xi_k \tag{150}$$

$$2E_k u_k^* v_k = \Delta \tag{151}$$

and additionally $2E_k u_k v_k^* = \Delta^*$. (By eliminating E_k you should be able to show that these equations are equivalent with $2\xi_k u_k v_k - \Delta u_k^2 + \Delta^* v_k^2 = 0$.) Adding the absolute squares of these two equations gives

$$E_k^2(|u_k|^2 + |v_k|^2)^2 = \xi_k^2 + |\Delta|^2$$

But since $|u_k|^2 + |v_k|^2 = 1$ due to the transformation being canonical, we have

$$E_k = \sqrt{\xi_k^2 + |\Delta|^2}$$

The constants Ω_0 and C can be identified as before. Notice that there was not need to assume a real Δ here. We have the full solution right away. It is also clear how the phases of u_k and v_k are related to the phase of Δ. If we write $\Delta = |\Delta|e^{i\phi}$, $u_k = |u_k|e^{i\phi_u}$, and $v_k = |v_k|e^{i\phi_v}$, then from the second of (150) we clearly have $\phi_v - \phi_u = \phi + 2\pi n$, where we can choose $n = 0$. Choosing also $\phi_u = 0$, we have $\phi_v = \phi$.