1. **Grand potential for a superconductor**
Calculate the grand potential $\Omega = -\ln[\mathrm{Tr}\, e^{-\beta\check{K}}]/\beta$ for $\check{K}_{\text{eff}} = \sum_{\boldsymbol{k},\sigma} E_k \check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma} + \Omega_0$ to obtain the result

$$\Omega = \Omega_0 - \frac{2}{\beta}\sum_{\boldsymbol{k}} \ln\left(1 + e^{-\beta E_k}\right).$$

(Hint: Calculate the trace $\mathrm{Tr}[\cdots] = \sum_{\{n_{\boldsymbol{k}\sigma}\}=0}^{1}\langle\{n_{\boldsymbol{k}\sigma}\}|\cdots|\{n_{\boldsymbol{k}\sigma}\}\rangle$ in the basis of the γ number operator $\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma}|\{n_{\boldsymbol{k}'\sigma'}\}\rangle = n_{\boldsymbol{k}\sigma}|\{n_{\boldsymbol{k}'\sigma'}\}\rangle$.)

Solution:
We have to calculate $\mathrm{Tr}\, e^{-\beta\check{K}_{\text{eff}}}$, where $\check{K}_{\text{eff}} = \sum_{\boldsymbol{k},\sigma} E_k \check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma} + \Omega_0$. As hinted, it is easiest to calculate the trace $\mathrm{Tr}[\cdots] = \sum_{\{n_{\boldsymbol{k}\sigma}\}=0}^{1}\langle\{n_{\boldsymbol{k}\sigma}\}|\cdots|\{n_{\boldsymbol{k}\sigma}\}\rangle$ in the basis of the γ number operator $\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma}|\{n_{\boldsymbol{k}'\sigma'}\}\rangle = n_{\boldsymbol{k}\sigma}|\{n_{\boldsymbol{k}'\sigma'}\}\rangle$. This is because the effective Hamiltonian $\check{K}_{\text{eff}}$ is diagonal in this basis. Now,

$$\begin{aligned}
e^{-\beta\Omega} &= \mathrm{Tr}\, e^{-\beta\check{K}_{\text{eff}}} = e^{-\beta\Omega_0}\,\mathrm{Tr}\exp\left(-\beta\sum_{\boldsymbol{k},\sigma} E_k\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma}\right)\\
&= e^{-\beta\Omega_0}\sum_{\{n_{\boldsymbol{k}\sigma}\}=0}^{1}\langle\{n_{\boldsymbol{k}\sigma}\}|\exp\left(-\beta\sum_{\boldsymbol{k},\sigma} E_k\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma}\right)|\{n_{\boldsymbol{k}\sigma}\}\rangle\\
&= e^{-\beta\Omega_0}\sum_{\{n_{\boldsymbol{k}\sigma}\}=0}^{1}\langle\{n_{\boldsymbol{k}\sigma}\}|\prod_{\boldsymbol{k},\sigma}\exp\left(-\beta E_k\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}\check{\gamma}_{\boldsymbol{k}\sigma}\right)|\{n_{\boldsymbol{k}\sigma}\}\rangle\\
&= e^{-\beta\Omega_0}\sum_{\{n_{\boldsymbol{k}\sigma}\}=0}^{1}\langle\{n_{\boldsymbol{k}\sigma}\}|\prod_{\boldsymbol{k},\sigma}\exp\left(-\beta E_k n_{\boldsymbol{k}\sigma}\right)|\{n_{\boldsymbol{k}\sigma}\}\rangle\\
&= e^{-\beta\Omega_0}\sum_{\{n_{\boldsymbol{k}\sigma}\}=0}^{1}\prod_{\boldsymbol{k},\sigma}\exp\left(-\beta E_k n_{\boldsymbol{k}\sigma}\right)\langle\{n_{\boldsymbol{k}\sigma}\}|\{n_{\boldsymbol{k}\sigma}\}\rangle\\
&= e^{-\beta\Omega_0}\sum_{\{n_{\boldsymbol{k}\sigma}\}=0}^{1}\prod_{\boldsymbol{k},\sigma}\exp\left(-\beta E_k n_{\boldsymbol{k}\sigma}\right)\\
&= e^{-\beta\Omega_0}\sum_{n_{\boldsymbol{k}_1\sigma_1}=0}^{1}\exp\left(-\beta E_{k_1} n_{\boldsymbol{k}_1\sigma_1}\right)\sum_{n_{\boldsymbol{k}_2\sigma_2}=0}^{1}\exp\left(-\beta E_{k_2} n_{\boldsymbol{k}_2\sigma_2}\right)\dots\\
&= e^{-\beta\Omega_0}\prod_{\boldsymbol{k},\sigma}\sum_{n_{\boldsymbol{k}\sigma}=0}^{1}\exp\left(-\beta E_k n_{\boldsymbol{k}\sigma}\right)
\end{aligned}$$

$$= e^{-\beta\Omega_0} \prod_{\boldsymbol{k},\sigma} [\exp(-\beta E_k \cdot 0) + \exp(-\beta E_k \cdot 1)]$$
$$= e^{-\beta\Omega_0} \prod_{\boldsymbol{k},\sigma} [1 + \exp(-\beta E_k)]\,.$$

From this we calculate

$$\begin{aligned}
\Omega &= -\ln[\mathrm{Tr}\, e^{-\beta \check{K}}]/\beta \\
&= -\frac{1}{\beta} \ln \left\{ e^{-\beta\Omega_0} \prod_{\boldsymbol{k},\sigma} [1 + \exp(-\beta E_k)] \right\} \\
&= -\frac{1}{\beta} \left\{ -\beta\Omega_0 + \ln \prod_{\boldsymbol{k},\sigma} [1 + \exp(-\beta E_k)] \right\} \\
&= \Omega_0 - \frac{1}{\beta} \sum_{\boldsymbol{k},\sigma} \ln [1 + \exp(-\beta E_k)] \\
&= \Omega_0 - \frac{2}{\beta} \sum_{\boldsymbol{k}} \ln \left(1 + e^{-\beta E_k}\right).
\end{aligned} \tag{1}$$

In the last step we summed over spin, and noted that the summand does not depend on spin.

2. **Gap equation: derivation**
Do all the intermediate steps in deriving the gap equation [(163)] from the definition

$$\Delta = \frac{g}{L^3} \sum_{\boldsymbol{k}} \langle \check{a}_{-\boldsymbol{k}\downarrow} \check{a}_{\boldsymbol{k}\uparrow} \rangle.$$

(Hint: Make use of the previous exercise to show $\langle \check{\gamma}^\dagger_{\boldsymbol{k}\sigma} \check{\gamma}_{\boldsymbol{k}\sigma} \rangle = n(E_k)$, where $n(E) = 1/(e^{\beta E} + 1)$, and $\langle \check{\gamma}_{\boldsymbol{k}\sigma} \check{\gamma}_{\boldsymbol{k}'\sigma'} \rangle = \langle \check{\gamma}^\dagger_{\boldsymbol{k}\sigma} \check{\gamma}^\dagger_{\boldsymbol{k}'\sigma'} \rangle = 0.$)

Solution:
We have to calculate

$$\langle \check{a}_{-\mathbf{k}\downarrow} \check{a}_{\mathbf{k}\uparrow} \rangle = \frac{1}{e^{-\beta\Omega}} \mathrm{Tr}(\check{a}_{-\mathbf{k}\downarrow} \check{a}_{\mathbf{k}\uparrow} e^{-\beta K_{eff}}),$$

where $e^{-\beta\Omega} = \mathrm{Tr}(e^{-\beta K_{eff}})$. It is again easiest to calculate the trace in the eigenbasis of the number operator $\check{\gamma}^\dagger_{\boldsymbol{k}\sigma} \check{\gamma}_{\boldsymbol{k}\sigma}$, $\check{\gamma}^\dagger_{\boldsymbol{k}\sigma} \check{\gamma}_{\boldsymbol{k}\sigma} |\{n_{\boldsymbol{k}'\sigma'}\}\rangle = n_{\boldsymbol{k}\sigma} |\{n_{\boldsymbol{k}'\sigma'}\}\rangle$.

Now, using

$$\begin{aligned}
\check{a}_{-\mathbf{k}\downarrow} &= u_k^* \check{\gamma}_{-\mathbf{k}\downarrow} - v_k \check{\gamma}^\dagger_{\mathbf{k}\uparrow} \\
\check{a}_{\mathbf{k}\uparrow} &= u_k^* \check{\gamma}_{\mathbf{k}\uparrow} + v_k \check{\gamma}^\dagger_{-\mathbf{k}\downarrow}
\end{aligned}$$

we have

$$
\begin{aligned}
\check{a}_{-\mathbf{k}\downarrow}\check{a}_{\mathbf{k}\uparrow} &= (u_k^*\check{\gamma}_{-\mathbf{k}\downarrow} - v_k\check{\gamma}_{\mathbf{k}\uparrow}^\dagger)(u_k^*\check{\gamma}_{\mathbf{k}\uparrow} + v_k\check{\gamma}_{-\mathbf{k}\downarrow}^\dagger) \\
&= u_k^* v_k(1 - \gamma_{\mathbf{k}\uparrow}^\dagger\gamma_{\mathbf{k}\uparrow} - \gamma_{-\mathbf{k}\downarrow}^\dagger\gamma_{-\mathbf{k}\downarrow}) + (u_k^*)^2\check{\gamma}_{-\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\uparrow} - v_k^2\check{\gamma}_{\mathbf{k}\uparrow}^\dagger\check{\gamma}_{-\mathbf{k}\downarrow}^\dagger
\end{aligned}
$$

where we used one of the anticommutation relations. Thus the expectation value is

$$
\langle\check{a}_{-\mathbf{k}\downarrow}\check{a}_{\mathbf{k}\uparrow}\rangle = u_k^* v_k(1 - \langle\gamma_{\mathbf{k}\uparrow}^\dagger\gamma_{\mathbf{k}\uparrow}\rangle - \langle\gamma_{-\mathbf{k}\downarrow}^\dagger\gamma_{-\mathbf{k}\downarrow}\rangle) + (u_k^*)^2\langle\check{\gamma}_{-\mathbf{k}\downarrow}\check{\gamma}_{\mathbf{k}\uparrow}\rangle - v_k^2\langle\check{\gamma}_{\mathbf{k}\uparrow}^\dagger\check{\gamma}_{-\mathbf{k}\downarrow}^\dagger\rangle.
$$

In exercise 1 we calculated $e^{-\beta\Omega} = e^{-\beta\Omega_0}\prod_{\boldsymbol{k},\sigma}\left[1 + \exp\left(-\beta E_k\right)\right]$. Following the same procedure we have

$$
\begin{aligned}
\langle\check{\gamma}_{\boldsymbol{k}\sigma}^\dagger\check{\gamma}_{\boldsymbol{k}\sigma}\rangle &= \frac{1}{e^{-\beta\Omega}}\mathrm{Tr}(\check{\gamma}_{\boldsymbol{k}\sigma}^\dagger\check{\gamma}_{\boldsymbol{k}\sigma}e^{-\beta K_{eff}}) \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}}\,\mathrm{Tr}\left[\check{\gamma}_{\boldsymbol{k}\sigma}^\dagger\check{\gamma}_{\boldsymbol{k}\sigma}\exp\left(-\beta\sum_{\boldsymbol{k}',\sigma'}E_{k'}\check{\gamma}_{\boldsymbol{k}'\sigma'}^\dagger\check{\gamma}_{\boldsymbol{k}'\sigma'}\right)\right] \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}}\sum_{\{n_{\boldsymbol{k}''\sigma''}\}=0}^{1}\langle\{n_{\boldsymbol{k}''\sigma''}\}|\check{\gamma}_{\boldsymbol{k}\sigma}^\dagger\check{\gamma}_{\boldsymbol{k}\sigma}\exp\left(-\beta\sum_{\boldsymbol{k}',\sigma'}E_{k'}\check{\gamma}_{\boldsymbol{k}'\sigma'}^\dagger\check{\gamma}_{\boldsymbol{k}'\sigma'}\right)|\{n_{\boldsymbol{k}''\sigma''}\}\rangle \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}}\sum_{\{n_{\boldsymbol{k}''\sigma''}\}=0}^{1}\langle\{n_{\boldsymbol{k}''\sigma''}\}|n_{\boldsymbol{k}\sigma}\prod_{\boldsymbol{k}',\sigma'}\exp\left(-\beta E_{k'}n_{\boldsymbol{k}'\sigma'}\right)|\{n_{\boldsymbol{k}''\sigma''}\}\rangle \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}}\sum_{\{n_{\boldsymbol{k}''\sigma''}\}=0}^{1}n_{\boldsymbol{k}\sigma}\prod_{\boldsymbol{k}',\sigma'}\exp\left(-\beta E_{k'}n_{\boldsymbol{k}'\sigma'}\right)\underbrace{\langle\{n_{\boldsymbol{k}''\sigma''}\}|\{n_{\boldsymbol{k}''\sigma''}\}\rangle}_{=1} \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}}\sum_{\{n_{\boldsymbol{k}''\sigma''}\}=0}^{1}n_{\boldsymbol{k}\sigma}\prod_{\boldsymbol{k}',\sigma'}\exp\left(-\beta E_{k'}n_{\boldsymbol{k}'\sigma'}\right) \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}}\sum_{n_{\boldsymbol{k}\sigma}=0}^{1}n_{\boldsymbol{k}\sigma}\exp\left(-\beta E_k n_{\boldsymbol{k}\sigma}\right)\prod_{\boldsymbol{k}',\sigma'\neq\boldsymbol{k},\sigma}\;\sum_{n_{\boldsymbol{k}'\sigma'}=0}^{1}\exp\left(-\beta E_{k'}n_{\boldsymbol{k}'\sigma'}\right) \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}}\exp\left(-\beta E_k\right)\prod_{\boldsymbol{k}',\sigma'\neq\boldsymbol{k},\sigma}\left[1 + \exp\left(-\beta E_{k'}\right)\right] \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega_0}\prod_{\boldsymbol{k}'',\sigma''}\left[1 + \exp\left(-\beta E_k''\right)\right]}\cdot\exp\left(-\beta E_k\right)\prod_{\boldsymbol{k}',\sigma'\neq\boldsymbol{k},\sigma}\left[1 + \exp\left(-\beta E_{k'}\right)\right] \\
&= \frac{\prod_{\boldsymbol{k}',\sigma'\neq\boldsymbol{k},\sigma}\left[1 + \exp\left(-\beta E_{k'}\right)\right]}{\prod_{\boldsymbol{k}'',\sigma''}\left[1 + \exp\left(-\beta E_k''\right)\right]}\cdot\exp\left(-\beta E_k\right) \\
&= \frac{1}{1 + \exp\left(-\beta E_k\right)}\cdot\exp\left(-\beta E_k\right) \\
&= \frac{1}{1 + \exp\left(\beta E_k\right)}
\end{aligned}
$$

$$= n(E_k),$$

while

$$\begin{aligned}
\langle \check{\gamma}_{\boldsymbol{k}\sigma} \check{\gamma}_{\tilde{\boldsymbol{k}}\tilde{\sigma}} \rangle &= \frac{1}{e^{-\beta\Omega}} \mathrm{Tr}(\check{\gamma}_{\boldsymbol{k}\sigma} \check{\gamma}_{\tilde{\boldsymbol{k}}\tilde{\sigma}} e^{-\beta K_{eff}}) \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}} \mathrm{Tr}\left[\check{\gamma}_{\boldsymbol{k}\sigma} \check{\gamma}_{\tilde{\boldsymbol{k}}\tilde{\sigma}} \exp\left(-\beta \sum_{\boldsymbol{k}',\sigma'} E_{k'} \check{\gamma}^\dagger_{\boldsymbol{k}'\sigma'} \check{\gamma}_{\boldsymbol{k}'\sigma'} \right) \right] \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}} \sum_{\{n_{\boldsymbol{k}''\sigma''}\}=0}^{1} \langle \{n_{\boldsymbol{k}''\sigma''}\} | \check{\gamma}_{\boldsymbol{k}\sigma} \check{\gamma}_{\tilde{\boldsymbol{k}}\tilde{\sigma}} \exp\left(-\beta \sum_{\boldsymbol{k}',\sigma'} E_{k'} \check{\gamma}^\dagger_{\boldsymbol{k}'\sigma'} \check{\gamma}_{\boldsymbol{k}'\sigma'} \right) | \{n_{\boldsymbol{k}''\sigma''}\} \rangle \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}} \sum_{\{n_{\boldsymbol{k}''\sigma''}\}=0}^{1} \langle \{n_{\boldsymbol{k}''\sigma''}\} | \check{\gamma}_{\boldsymbol{k}\sigma} \check{\gamma}_{\tilde{\boldsymbol{k}}\tilde{\sigma}} \prod_{\boldsymbol{k}',\sigma'} \exp\left(-\beta E_{k'} n_{\boldsymbol{k}'\sigma'} \right) | \{n_{\boldsymbol{k}''\sigma''}\} \rangle \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}} \sum_{\{n_{\boldsymbol{k}''\sigma''}\}=0}^{1} \prod_{\boldsymbol{k}',\sigma'} \exp\left(-\beta E_{k'} n_{\boldsymbol{k}'\sigma'} \right) \underbrace{\langle \{n_{\boldsymbol{k}''\sigma''}\} | \check{\gamma}_{\boldsymbol{k}\sigma} \check{\gamma}_{\tilde{\boldsymbol{k}}\tilde{\sigma}} | \{n_{\boldsymbol{k}''\sigma''}\} \rangle}_{=0} \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle \check{\gamma}^\dagger_{\boldsymbol{k}\sigma} \check{\gamma}^\dagger_{\tilde{\boldsymbol{k}}\tilde{\sigma}} \rangle &= \dots \\
&= \frac{e^{-\beta\Omega_0}}{e^{-\beta\Omega}} \sum_{\{n_{\boldsymbol{k}''\sigma''}\}=0}^{1} \prod_{\boldsymbol{k}',\sigma'} \exp\left(-\beta E_{k'} n_{\boldsymbol{k}'\sigma'} \right) \underbrace{\langle \{n_{\boldsymbol{k}''\sigma''}\} | \check{\gamma}^\dagger_{\boldsymbol{k}\sigma} \check{\gamma}^\dagger_{\tilde{\boldsymbol{k}}\tilde{\sigma}} | \{n_{\boldsymbol{k}''\sigma''}\} \rangle}_{=0} \\
&= 0.
\end{aligned}$$

We now have

$$\begin{aligned}
\langle \check{a}_{-\mathbf{k}\downarrow} \check{a}_{\mathbf{k}\uparrow} \rangle &= u_k^* v_k (1 - \langle \gamma^\dagger_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\uparrow} \rangle - \langle \gamma^\dagger_{-\mathbf{k}\downarrow} \gamma_{-\mathbf{k}\downarrow} \rangle) + (u_k^*)^2 \langle \check{\gamma}_{-\mathbf{k}\downarrow} \check{\gamma}_{\mathbf{k}\uparrow} \rangle - v_k^2 \langle \check{\gamma}^\dagger_{\mathbf{k}\uparrow} \check{\gamma}^\dagger_{-\mathbf{k}\downarrow} \rangle \\
&= u_k^* v_k [1 - n(E_k) - n(E_k)] + (u_k^*)^2 \cdot 0 - v_k^2 \cdot 0 \\
&= u_k^* v_k \left[1 - 2n(E_k) \right].
\end{aligned}$$

Next, we assume Δ etc. to be real. Then using

$$u_k v_k = \sqrt{\frac{1}{4}(1 - \frac{\xi_k^2}{E_k^2})} = \frac{1}{2} \frac{\Delta}{E_k}$$

and

$$1 - 2\frac{1}{e^x + 1} = \frac{e^x + 1 - 2}{e^x + 1} = \frac{e^x - 1}{e^x + 1} = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \tanh \frac{x}{2}$$

with $x = \beta E_k$, we have

$$\Delta = \frac{g}{2L^3} \sum_{\mathbf{k}} \frac{\Delta}{E_k} \tanh \frac{E_k}{2k_B T}$$
$$1 = \frac{g}{2L^3} \sum_{\mathbf{k}} \frac{1}{E_k} \tanh \frac{E_k}{2k_B T}$$

The using (74), (76), and introducing a cutoff

$$1 = \frac{gN(0)}{2} \int_{-\epsilon_c}^{\epsilon_c} \frac{1}{\sqrt{\xi_k^2 + \Delta^2}} \tanh \frac{\sqrt{\xi_k^2 + \Delta^2}}{2k_B T} d\xi_k$$
$$\frac{1}{gN(0)} = \int_0^{\epsilon_c} \frac{1}{\sqrt{\xi_k^2 + \Delta^2}} \tanh \frac{\sqrt{\xi_k^2 + \Delta^2}}{2k_B T} d\xi_k$$

Where in the final step the symmetry of the integrand was used. This is the gap equation.

3. **Hartree-Fock interaction**
Show that the Hartree-Fock (not anomalous) interaction energy $\langle \check{V}_{\mathrm{HF}} \rangle$ is the same for normal and superconducting states. This demonstrates *a posteriori* that the neglect of non-anomalous HF terms in the treatment of the superconducting state is allowed. The HF potential energy for a spin-conserving contact interaction can be written as

$$\check{V}_{\mathrm{HF}} = \frac{g}{L^3} \sum_{\boldsymbol{k}_1} \sum_{\boldsymbol{k}_2} \sum_{\boldsymbol{k}_3} \sum_{\boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2, \boldsymbol{k}_3+\boldsymbol{k}_4} \big(\langle \check{a}^\dagger_{\boldsymbol{k}_3\uparrow} \check{a}_{\boldsymbol{k}_1\uparrow} \rangle \check{a}^\dagger_{\boldsymbol{k}_4\downarrow} \check{a}_{\boldsymbol{k}_2\downarrow} + \langle \check{a}^\dagger_{\boldsymbol{k}_4\downarrow} \check{a}_{\boldsymbol{k}_2\downarrow} \rangle \check{a}^\dagger_{\boldsymbol{k}_3\uparrow} \check{a}_{\boldsymbol{k}_1\uparrow} - \langle \check{a}^\dagger_{\boldsymbol{k}_4\downarrow} \check{a}_{\boldsymbol{k}_2\downarrow} \rangle \langle \check{a}^\dagger_{\boldsymbol{k}_3\uparrow} \check{a}_{\boldsymbol{k}_1\uparrow} \rangle \big).$$

(Hint: as intermediate results show that $\langle \check{a}^\dagger_{\boldsymbol{k}_3\uparrow} \check{a}_{\boldsymbol{k}_1\uparrow} \rangle = \delta_{\boldsymbol{k}_3, \boldsymbol{k}_1} C_{k_1}$ and $\langle \check{V}_{\mathrm{HF}} \rangle = (g/L^3)(\sum_{\boldsymbol{k}} C_k)^2$ and deduce that the sum for $|\xi_k| < \epsilon_c$ is independent of Δ.)

Solution:

The main idea here is to show that the Hartree-Fock energy in the superconducting state is independent of Δ, which, in the normal state, is zero.

The expectation value of the HF potential energy is

$$\langle \check{V}_{\mathrm{HF}} \rangle = \frac{g}{L^3} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_2, \mathbf{k}_3+\mathbf{k}_4} \big(\langle \check{a}^\dagger_{\mathbf{k}_3\uparrow} \check{a}_{\mathbf{k}_1\uparrow} \rangle \langle \check{a}^\dagger_{\mathbf{k}_4\downarrow} \check{a}_{\mathbf{k}_2\downarrow} \rangle + \langle \check{a}^\dagger_{\mathbf{k}_4\downarrow} \check{a}_{\mathbf{k}_2\downarrow} \rangle \langle \check{a}^\dagger_{\mathbf{k}_3\uparrow} \check{a}_{\mathbf{k}_1\uparrow} \rangle - \langle \check{a}^\dagger_{\mathbf{k}_4\downarrow} \check{a}_{\mathbf{k}_2\downarrow} \rangle \langle \check{a}^\dagger_{\mathbf{k}_3\uparrow} \check{a}_{\mathbf{k}_1\uparrow} \rangle \big).$$

The last two terms cancel to yield:

$$\langle \check{V}_{\mathrm{HF}} \rangle = \frac{g}{L^3} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_3} \sum_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_2, \mathbf{k}_3+\mathbf{k}_4} \langle \check{a}^\dagger_{\mathbf{k}_3\uparrow} \check{a}_{\mathbf{k}_1\uparrow} \rangle \langle \check{a}^\dagger_{\mathbf{k}_4\downarrow} \check{a}_{\mathbf{k}_2\downarrow} \rangle$$

Now the idea is again to use the Bogoliubov transformation to express a operators in terms of γ operators:

$$\breve{a}_{\mathbf{k}\uparrow} = u_k^* \breve{\gamma}_{\mathbf{k}\uparrow} + v_k \breve{\gamma}^\dagger_{-\mathbf{k}\downarrow} \qquad \breve{a}_{\mathbf{k}\downarrow} = u_k^* \breve{\gamma}_{\mathbf{k}\downarrow} - v_k \breve{\gamma}^\dagger_{-\mathbf{k}\uparrow} \tag{2}$$

$$\breve{a}^\dagger_{\mathbf{k}\uparrow} = u_k \breve{\gamma}^\dagger_{\mathbf{k}\uparrow} + v_k^\star \breve{\gamma}_{-\mathbf{k}\downarrow} \qquad \breve{a}^\dagger_{\mathbf{k}\downarrow} = u_k \breve{\gamma}^\dagger_{\mathbf{k}\downarrow} - v_k^\star \breve{\gamma}_{-\mathbf{k}\uparrow}. \tag{3}$$

and use the fact that expectation values of the type $\langle\gamma\gamma\rangle$ and $\langle\gamma^\dagger\gamma^\dagger\rangle$ vanish. Thus we have

$$\langle \breve{a}^\dagger_{\mathbf{k}_3\uparrow}\breve{a}_{\mathbf{k}_1\uparrow}\rangle = \delta_{\mathbf{k}_3\mathbf{k}_1}[|u_{k_1}|^2 n(E_{k_1}) - |v_{k_1}|^2 n(E_{k_1}) + |v_{k_1}|^2]$$

and

$$\langle \breve{a}^\dagger_{\mathbf{k}_4\downarrow}\breve{a}_{\mathbf{k}_2\downarrow}\rangle = \delta_{\mathbf{k}_2\mathbf{k}_4}[|u_{k_2}|^2 n(E_{k_2}) - |v_{k_2}|^2 n(E_{k_2}) + |v_{k_2}|^2]$$

where we used the commutation relations and the previously derived results for the Fermi funtion $n(E)$.

Here

$$\begin{aligned}
(|u_k|^2 - |v_k|^2)n(E_k) + v_k^2 &= \left[\frac{1}{2}(1-\frac{\xi_k}{E_k}) - \frac{1}{2}(1-\frac{\xi_k}{E_k})\right]\frac{1}{e^{\beta E_k}+1} + \frac{1}{2}\left(1-\frac{\xi_k}{E_k}\right) \\
&= \frac{\xi_k/E_k}{e^{\beta E_k}+1} - \frac{\xi_k/E_k}{2} + \frac{1}{2} = \frac{\xi_k}{E_k}\left(\frac{2-e^{\beta E_k}-1}{2(e^{\beta E_k}+1)}\right) + \frac{1}{2} \\
&= \frac{1}{2} - \frac{\xi_k}{2E_k}\left(\frac{e^{\beta E_k}-1}{e^{\beta E_k}+1}\right) = \frac{1}{2} - \frac{\xi_k}{2E_k}\tanh\frac{E_k}{2k_BT}
\end{aligned}$$

Thus

$$\begin{aligned}
\langle \breve{V}_{\mathrm{HF}}\rangle &= \frac{g}{L^3}\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}\sum_{\mathbf{k}_3}\sum_{\mathbf{k}_4}\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4}\delta_{\mathbf{k}_1,\mathbf{k}_3}\delta_{\mathbf{k}_2,\mathbf{k}_4} \\
&\quad\times\left(\frac{1}{2} - \frac{\xi_{k_1}}{2E_{k_1}}\tanh\frac{E_{k_1}}{2k_BT}\right)\left(\frac{1}{2} - \frac{\xi_{k_2}}{2E_{k_2}}\tanh\frac{E_{k_2}}{2k_BT}\right) \\
&= \frac{g}{L^3}\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}\left(\frac{1}{2} - \frac{\xi_{k_1}}{2E_{k_1}}\tanh\frac{E_{k_1}}{2k_BT}\right)\left(\frac{1}{2} - \frac{\xi_{k_2}}{2E_{k_2}}\tanh\frac{E_{k_2}}{2k_BT}\right) \\
&= \frac{g}{2L^3}\left[\sum_{\mathbf{k}_1}\left(\frac{1}{2} - \frac{\xi_{k_1}}{2E_{k_1}}\tanh\frac{E_{k_1}}{2k_BT}\right)\right]^2 \\
&= \frac{g}{L^3}\left[\frac{N(0)L^3}{2}\int_{-\epsilon_c}^{\epsilon_c}\left(1 - \frac{\xi_{k_1}}{\sqrt{\xi_k^2+\Delta^2}}\tanh\frac{\sqrt{\xi_k^2+\Delta^2}}{2k_BT}\right)\right]^2
\end{aligned}$$

The second term in the integrand is antisymmetric and gives zero. Therefore

$$\langle \breve{V}_{\mathrm{HF}}\rangle = \frac{g}{L^3}\left[\frac{N(0)L^3}{2}2\epsilon_c\right]^2 = \frac{g}{L^3}\left[N(0)\right]^2 L^6\epsilon_c^2 = g\left[N(0)\right]^2\epsilon_c^2 L^3$$

which is independent of Δ, and thus must be the same for superconducting and normal states. Indeed, if one would set $\Delta = 0$ in the beginning and modify u_k and v_k accordingly, the calculation would fully correspond to the normal-state calculation. Note that this result is correctly proportional to the volume L^3 and has units of energy.

4. **Gap equation at $T = T_c$**
Prove the result
$$\int_0^{\epsilon_c} d\xi \, \frac{1}{\xi} \tanh \frac{\xi}{2k_B T_c} = \ln \frac{2\epsilon_c e^{\gamma}}{\pi k_B T_c},$$
where γ is a certain constant given by the definite integral:
$$\int_0^{\infty} \mathrm{d}x \, \frac{\ln x}{\cosh^2 x} = -\gamma + \ln \frac{\pi}{4}.$$

(Hint: use $\xi^{-1} = d(\ln \xi)/d\xi$ and integrate by parts. Verify that the integral converges, so that you can take the limit $\epsilon_c \to \infty$.)

Solution:

Consider the integral
$$\int_0^{\epsilon_c} d\xi \, \frac{1}{\xi} \tanh \frac{\xi}{2k_B T_c}$$
Change of variable: $x = \xi/(2k_B T_c)$, $d\xi = 2k_B T_c dx$, $1/\xi = 1/(2k_B T_c x)$. Now $\epsilon_c \gg 2k_B T_c$ implies $\tilde{\epsilon}_c = \epsilon_c/(2k_B T_c) \gg 1$.
$$\int_0^{\epsilon_c} d\xi \, \frac{1}{\xi} \tanh \frac{\xi}{2k_B T_c} = \int_0^{\tilde{\epsilon}_c} dx \frac{1}{x} \tanh x = |_0^{\tilde{\epsilon}_c} \ln x \tanh x - \int_0^{\tilde{\epsilon}_c} dx \ln x (\frac{d}{dx} \tanh x)$$
Here $\frac{d}{dx} \tanh x = 1/\cosh^2 x \sim 4e^{-2x}$, so we can take $\tilde{\epsilon}_c \to \infty$ in the integral and it still converges. In the first term the limit $x \to 0$ converges because $\ln x \tanh x \sim x \ln x \to 0$. The term $\ln \tilde{\epsilon}_c \tanh \tilde{\epsilon}_c \approx \ln \tilde{\epsilon}_c$ must be kept, however. So
$$\int_0^{\epsilon_c} d\xi \, \frac{1}{\xi} \tanh \frac{\xi}{2k_B T_c} = \ln \tilde{\epsilon}_c - \int_0^{\infty} dx \frac{\ln x}{\cosh^2 x}$$
Here the remaining integral can be done with Mathematica, for example, giving
$$\int_0^{\epsilon_c} d\xi \, \frac{1}{\xi} \tanh \frac{\xi}{2k_B T_c} = \ln \tilde{\epsilon}_c + \gamma - \ln(\frac{\pi}{4})$$
where γ is the Euler-Mascheroni constant, defined as $\gamma = \lim_{n\to\infty}[\sum_{k=1}^{n} \frac{1}{k} - \ln(n)] \approx 0.5772$. Thus
$$\int_0^{\epsilon_c} d\xi \, \frac{1}{\xi} \tanh \frac{\xi}{2k_B T_c} = \ln \frac{4\tilde{\epsilon}_c e^{\gamma}}{\pi} = \ln \frac{2\epsilon_c e^{\gamma}}{\pi k_B T_c}$$

which was to be shown.

5. **Gap equation: elimination of $gN(0)$ and ϵ_c**
Show that the gap equation (in weak coupling approximation $gN(0) \ll 1$) can be written in the form

$$\ln \frac{T_c}{T} = \int_0^\infty \left(\frac{\tanh(\xi/2k_BT)}{\xi} - \frac{\tanh(\sqrt{\xi^2+\Delta^2}/2k_BT)}{\sqrt{\xi^2+\Delta^2}} \right) d\xi.$$

Verify that the integral converges, so that it is possible to put $\epsilon_c \to \infty$. In this way the two parameters $gN(0)$ and ϵ_c have been replaced by a single one: T_c.

Solution:

The gap equation is

$$\frac{1}{gN(0)} = \int_0^{\epsilon_c} d\xi \frac{1}{\sqrt{\xi^2+\Delta^2}} \tanh \frac{\sqrt{\xi^2+\Delta^2}}{2k_BT}.$$

At $T = T_c$ we know that $\Delta = 0$, and so

$$\frac{1}{gN(0)} = \int_0^{\epsilon_c} d\xi \frac{1}{\xi} \tanh \frac{\xi}{2k_BT_c}.$$

This integral was calculated previously:

$$\int_0^{\epsilon_c} d\xi \frac{1}{\xi} \tanh \frac{\xi}{2k_BT_c} = \ln \frac{2\epsilon_c e^\gamma}{\pi k_B T_c}.$$

Now, consider the same integral, but with $T_c \to T$:

$$\int_0^{\epsilon_c} d\xi \frac{1}{\xi} \tanh \frac{\xi}{2k_BT} = \ln \frac{2\epsilon_c e^\gamma}{\pi k_B T}.$$

By comparing these two integrals we see that

$$\begin{aligned} \int_0^{\epsilon_c} d\xi \frac{1}{\xi} \tanh \frac{\xi}{2k_BT_c} &= \int_0^{\epsilon_c} d\xi \frac{1}{\xi} \tanh \frac{\xi}{2k_BT} + \ln \frac{2\epsilon_c e^\gamma}{\pi k_B T_c} - \ln \frac{2\epsilon_c e^\gamma}{\pi k_B T} \\ &= \int_0^{\epsilon_c} d\xi \frac{1}{\xi} \tanh \frac{\xi}{2k_BT} - \ln \frac{T_c}{T}. \end{aligned}$$

Thus we may write the gap equation at $T = T_c$ as

$$\frac{1}{gN(0)} = \int_0^{\epsilon_c} d\xi \frac{1}{\xi} \tanh \frac{\xi}{2k_BT} - \ln \frac{T_c}{T}.$$

Subtracting the general gap equation from this, we have

$$\int_0^{\epsilon_c} d\xi[\frac{1}{\xi}\tanh\frac{\xi}{2k_BT} - \frac{1}{\sqrt{\xi^2+\Delta^2}}\tanh\frac{\sqrt{\xi^2+\Delta^2}}{2k_BT}] = \ln\frac{T_c}{T}.$$

In the limit $\epsilon_c \gg k_BT_c$ the integrand can get values $\xi \gg \Delta, k_BT$. In this limit we may replace the tanh-functions by 1 so that the integrand is of the form

$$\frac{1}{\xi} - \frac{1}{\sqrt{\xi^2+\Delta^2}} = \frac{1}{\xi} - \frac{1}{\xi}\left\{1 - \frac{1}{2}(\Delta/\xi)^2 + O[(\Delta/\xi)^4]\right\} = \frac{1}{2}\frac{\Delta^2}{\xi^3} + O[(\Delta/\xi)^5]$$

Clearly, the integral of this converges in the upper limit even if we take $\epsilon_c \to \infty$, and this is a good approximation if $\epsilon_c \gg k_BT_c$.