1. **BCS ground state**
Show that $\check{\gamma}_{\boldsymbol{k}\sigma}|\psi_0\rangle = 0$ for the BCS ground state $|\psi_0\rangle$, which means that $|\psi_0\rangle$ is the vacuum state for excitations. Consider at least the case $\sigma = \uparrow$.

(Hint: It is useful to define $c_{\boldsymbol{k}} = u_k + v_k a^\dagger_{\boldsymbol{k}\uparrow} a^\dagger_{-\boldsymbol{k}\downarrow}$ and to show that $[c_{\boldsymbol{k}}, c_{\boldsymbol{k}'}] = 0$.)

Solution:

Some preliminaries first. These should be useful also elsewhere. The BCS ground state is

$$|\psi_0\rangle = \prod_{\mathbf{k}} (u_k + v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow})|vac\rangle$$

where $|vac\rangle$ is the vacuum state: $a_{\mathbf{k}\uparrow}|vac\rangle = 0$. Defining $c_{\mathbf{k}} = u_k + v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow}$, we may write

$$|\psi_0\rangle = \prod_{\mathbf{k}} c_{\mathbf{k}}|vac\rangle$$

Is the order of $c_{\mathbf{k}}$s in the product relevant, or can we freely commute them? Let's see. For $\mathbf{k} \neq \mathbf{l}$ we find

$$\begin{aligned}
c_{\mathbf{k}} c_{\mathbf{l}} &= (u_k + v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow})(u_l + v_l a^\dagger_{\mathbf{l}\uparrow} a^\dagger_{-\mathbf{l}\downarrow}) \\
&= u_k u_l + u_k v_l a^\dagger_{\mathbf{l}\uparrow} a^\dagger_{-\mathbf{l}\downarrow} + u_l v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} + v_k v_l a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} a^\dagger_{\mathbf{l}\uparrow} a^\dagger_{-\mathbf{l}\downarrow} \\
&= u_l u_k + u_l v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} + u_k v_l a^\dagger_{\mathbf{l}\uparrow} a^\dagger_{-\mathbf{l}\downarrow} + v_l v_k a^\dagger_{\mathbf{l}\uparrow} a^\dagger_{-\mathbf{l}\downarrow} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \\
&= c_{\mathbf{l}} c_{\mathbf{k}}
\end{aligned}$$

Thus the operators commute. (Getting to the second to last line requires anticommuting operators in the four-operator term 4 times ($a^\dagger_i a^\dagger_j = -a^\dagger_j a^\dagger_i$), which thus keeps the sign intact.)

Now to the problem itself. Using the above commutation result, we can isolate from the BCS ground state an arbitrary $c_{\mathbf{k}} = u_k + v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow}$ factor and bring it to the front of the product. Thus we need to calculate for example

$$\gamma_{\mathbf{k}\uparrow}|\psi_0\rangle = \gamma_{\mathbf{k}\uparrow} c_{\mathbf{k}} \prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle$$

where $\gamma_{\mathbf{k}\uparrow} = u_k a_{\mathbf{k}\uparrow} - v_k a^\dagger_{-\mathbf{k}\downarrow}$. Now an intermediate result.

$$\begin{aligned}
\gamma_{\mathbf{k}\uparrow} c_{\mathbf{k}} &= (u_k a_{\mathbf{k}\uparrow} - v_k a^\dagger_{-\mathbf{k}\downarrow})(u_k + v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow}) \\
&= u_k^2 a_{\mathbf{k}\uparrow} + u_k v_k a_{\mathbf{k}\uparrow} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} - u_k v_k a^\dagger_{-\mathbf{k}\downarrow} - v_k^2 a^\dagger_{-\mathbf{k}\downarrow} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \\
&= u_k^2 a_{\mathbf{k}\uparrow} + u_k v_k a^\dagger_{-\mathbf{k}\downarrow} - u_k v_k a^\dagger_{\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} - u_k v_k a^\dagger_{-\mathbf{k}\downarrow} + v_k^2 a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} a^\dagger_{-\mathbf{k}\downarrow} \\
&= u_k^2 a_{\mathbf{k}\uparrow} + u_k v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} = u_k c_{\mathbf{k}} a_{\mathbf{k}\uparrow}
\end{aligned}$$

Since there is no $c_{\mathbf{k}}$-factor in the product $\prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}$ (and $[a_{\mathbf{k}\uparrow}, c_{-\mathbf{k}}] = 0$, as you may check), we may move the $a_{\mathbf{k}\uparrow}$ operator all the way through:

$$\gamma_{\mathbf{k}\uparrow}|\psi_0\rangle = \gamma_{\mathbf{k}\uparrow} c_{\mathbf{k}} \prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle = u_k c_{\mathbf{k}} a_{\mathbf{k}\uparrow} \prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle = u_k c_{\mathbf{k}} \prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}} a_{\mathbf{k}\uparrow}|vac\rangle = 0$$

A similar proof can be given for $\gamma_{\mathbf{k}\downarrow}|\psi_0\rangle = 0$. One can for example isolate $c_{-\mathbf{k}}$ in front of the product and take it on from there.

Note: One can in fact construct the state $|\psi_0\rangle$ from the requirement that $\gamma_{\mathbf{k}\uparrow}|\psi_0\rangle = 0$. This property leads directly to the guess that $|\psi_0\rangle \propto \prod_{\mathbf{k},\sigma} \gamma_{\mathbf{k},\sigma}|vac\rangle$ where $a_{\mathbf{k},\sigma}|vac\rangle = 0$. One may first show that $\gamma_{\mathbf{k},\uparrow}\gamma_{-\mathbf{k},\downarrow}|vac\rangle = v_k(u_k + v_k a^\dagger_{\mathbf{k},\uparrow} a^\dagger_{-\mathbf{k},\downarrow})|vac\rangle$ and then argue that $|\psi_0\rangle \propto \prod_{\mathbf{k}}(\gamma_{\mathbf{k},\uparrow}\gamma_{-\mathbf{k},\downarrow})|vac\rangle \propto \prod_{\mathbf{k}}(u_k + v_k a^\dagger_{\mathbf{k},\uparrow} a^\dagger_{-\mathbf{k},\downarrow})|vac\rangle$. This is left as an additional exercise.

2. **Normalization of the BCS ground state**
Assuming that $\langle \text{vac}|\text{vac}\rangle = 1$, show that the BCS ground state $|\psi_0\rangle$ is normalized as $\langle\psi_0|\psi_0\rangle = 1$.

Solution:

Define (once more) the operators $c_{\mathbf{k}} = u_k + v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow}$, which satisfy $[c_{\mathbf{k}}, c_{\mathbf{l}}] = 0$. Then

$$\langle\psi_0|\psi_0\rangle = \langle vac| \prod_{\mathbf{k}} c^\dagger_{\mathbf{k}} \prod_{\mathbf{l}} c_{\mathbf{l}}|vac\rangle$$

Let us order the products as follows

$$\langle\psi_0|\psi_0\rangle = \langle vac| \prod_{\mathbf{m}\neq\mathbf{k}} c^\dagger_{\mathbf{m}} c^\dagger_{\mathbf{k}} c_{\mathbf{k}} \prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle$$

Here

$$\begin{aligned}
c^\dagger_{\mathbf{k}} c_{\mathbf{k}} &= (u_k^* + v_k^* a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow})(u_k + v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow}) \\
&= |u_k|^2 + u_k^* v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} + v_k^* u_k a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} + |v_k|^2 a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \\
&= |u_k|^2 + |v_k|^2 a_{-\mathbf{k}\downarrow} a^\dagger_{-\mathbf{k}\downarrow} - |v_k|^2 a_{-\mathbf{k}\downarrow} a^\dagger_{\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} + u_k^* v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} + v_k^* u_k a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} \\
&= 1 - |v_k|^2 a^\dagger_{-\mathbf{k}\downarrow} a_{-\mathbf{k}\downarrow} + |v_k|^2 a_{-\mathbf{k}\downarrow} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} + u_k^* v_k a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} + v_k^* u_k a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow}
\end{aligned}$$

where the anticommutation rules and $|u_k|^2 + |v_k|^2 = 1$ were used. The last four terms clearly all produce zeroes in $\langle\psi_0|\psi_0\rangle$. What thus remains is simply

$$\langle\psi_0|\psi_0\rangle = \langle vac|\prod_{\mathbf{m}\neq\mathbf{k}} c^\dagger_{\mathbf{m}} \prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle = \ldots = \langle vac|vac\rangle = 1$$

The dots mean that we can repeat the above procedure by isolating some other $c^\dagger_{\mathbf{m}}c_{\mathbf{m}}$ with $\mathbf{m} \neq \mathbf{k}$ in the center of the sandwich. When this is done for all wave vectors, all that remains is $\langle vac|vac\rangle$, which equals 1 by assumption.

3. **Excitations of BCS state**
Let $|\psi_0\rangle$ be the BCS ground state. Show that the excited states $\check{\gamma}^\dagger_{\boldsymbol{k}\sigma}|\psi_0\rangle$ are of the form where the single-particle state $\boldsymbol{k}\sigma$ (to which particles are created by $\check{a}^\dagger_{\boldsymbol{k}\sigma}$) is populated and $-\boldsymbol{k}-\sigma$ is empty. You can limit to the case $\sigma =\uparrow$.

Solution:

Define (again) the operators $c_{\mathbf{k}} = u_k + v_k a^\dagger_{\mathbf{k}\uparrow}a^\dagger_{-\mathbf{k}\downarrow}$, which satisfy $[c_{\mathbf{k}}, c_{\mathbf{l}}] = 0$. Now first

$$\gamma^\dagger_{\mathbf{k}\uparrow}|\psi_0\rangle = \gamma^\dagger_{\mathbf{k}\uparrow}c_{\mathbf{k}}\prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle$$

Then by using the anticommutation relations

$$\begin{aligned}
\gamma^\dagger_{\mathbf{k}\uparrow}c_{\mathbf{k}} &= (u_k^* a^\dagger_{\mathbf{k}\uparrow} - v_k^* a_{-\mathbf{k}\downarrow})(u_k + v_k a^\dagger_{\mathbf{k}\uparrow}a^\dagger_{-\mathbf{k}\downarrow}) \\
&= |u_k|^2 a^\dagger_{\mathbf{k}\uparrow} + u_k^* v_k a^\dagger_{\mathbf{k}\uparrow}a^\dagger_{\mathbf{k}\uparrow}a^\dagger_{-\mathbf{k}\downarrow} - v_k^* u_k a_{-\mathbf{k}\downarrow} - |v_k|^2 a_{-\mathbf{k}\downarrow}a^\dagger_{\mathbf{k}\uparrow}a^\dagger_{-\mathbf{k}\downarrow} \\
&= |u_k|^2 a^\dagger_{\mathbf{k}\uparrow} - v_k^* u_k a_{-\mathbf{k}\downarrow} + |v_k|^2 a^\dagger_{\mathbf{k}\uparrow}a_{-\mathbf{k}\downarrow}a^\dagger_{-\mathbf{k}\downarrow} \\
&= |u_k|^2 a^\dagger_{\mathbf{k}\uparrow} - v_k^* u_k a_{-\mathbf{k}\downarrow} + |v_k|^2 a^\dagger_{\mathbf{k}\uparrow} - |v_k|^2 a^\dagger_{\mathbf{k}\uparrow}a^\dagger_{-\mathbf{k}\downarrow}a_{-\mathbf{k}\downarrow} \\
&= a^\dagger_{\mathbf{k}\uparrow} - v_k^* c_{\mathbf{k}} a_{-\mathbf{k}\downarrow}
\end{aligned}$$

Note that we used $|u_k|^2 + |v_k|^2 = 1$. Now back to what we were doing:

$$\begin{aligned}
\gamma^\dagger_{\mathbf{k}\uparrow}|\psi_0\rangle &= \gamma^\dagger_{\mathbf{k}\uparrow}c_{\mathbf{k}}\prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle = a^\dagger_{\mathbf{k}\uparrow}\prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle - v_k^* c_{\mathbf{k}}a_{-\mathbf{k}\downarrow}\prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle \\
&= a^\dagger_{\mathbf{k}\uparrow}\prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle - v_k^*\prod_{\mathbf{l}} c_{\mathbf{l}}a_{-\mathbf{k}\downarrow}|vac\rangle = a^\dagger_{\mathbf{k}\uparrow}\prod_{\mathbf{l}\neq\mathbf{k}} c_{\mathbf{l}}|vac\rangle
\end{aligned}$$

In this state clearly the level $-\mathbf{k}\downarrow$ is empty, and $\mathbf{k}\uparrow$ is filled. A similar proof can be repeated for $\gamma^\dagger_{\mathbf{k}\downarrow}|\psi_0\rangle$.

4. **Energy functional**
Show that for the energy functional ($u_k^2 = 1 - v_k^2$, $\xi_k = \hbar^2k^2/2m - \mu$)

$$\begin{aligned}
\Omega(T, V, \mu, v_k, \Delta) =& 2\sum_{\boldsymbol{k}}(\xi_k v_k^2 - \Delta u_k v_k) + \frac{L^3}{g}\Delta^2 \\
& - 2k_BT\sum_{\boldsymbol{k}}\ln(1 + e^{-\sqrt{\xi_k^2+\Delta^2}/k_BT}),
\end{aligned}$$

the relations

$$\frac{\partial \Omega}{\partial v_k} = 0, \qquad \frac{\partial \Omega}{\partial \Delta} = 0,$$

are equivalent with the conditions (152) and (158) of the lecture notes.

Solution:

The energy functional is

$$\begin{aligned}\Omega(T, V, \mu, v_k, \Delta) =& 2\sum_{\mathbf{k}}(\xi_k v_k^2 - \Delta u_k v_k) + \frac{L^3}{g}\Delta^2 \\ & - 2k_B T \sum_{\mathbf{k}} \ln(1 + e^{-\sqrt{\xi_k^2+\Delta^2}/k_B T}),\end{aligned}$$

First,

$$\frac{\partial \Omega}{\partial v_k} = 2(2\xi_k v_k - \Delta u_k - \Delta v_k \frac{\partial u_k}{\partial v_k})$$

Here we note that $u_k = \sqrt{1 - v_k^2}$ and thus

$$\frac{\partial u_k}{\partial v_k} = -\frac{2v_k}{2\sqrt{1-v_k^2}} = -\frac{v_k}{u_k}$$

So

$$\begin{aligned}\frac{\partial \Omega}{\partial v_k} &= 0 \\ 2\xi_k v_k - \Delta u_k + \Delta v_k^2/u_k &= 0 \\ 2\xi_k u_k v_k - \Delta u_k^2 + \Delta v_k^2 &= 0\end{aligned}$$

This is equal to (152). (Note that, for notational simplicity, we've only considered Ω to be a function of a single v_k, but of course there is one for each $\mathbf{k}$.)

Second,

$$\frac{\partial \Omega}{\partial \Delta} = -2\sum_{\mathbf{k}} u_k v_k + 2\frac{L^3}{g}\Delta - 2k_B T \sum_{\mathbf{k}} \frac{e^{-E_k/k_B T}}{1 + e^{-E_k/k_B T}}(-\frac{1}{k_B T})\frac{\partial E_k}{\partial \Delta}$$

Here we use $E_k = \sqrt{\xi_k^2 + \Delta^2}$ so that

$$\frac{\partial E_k}{\partial \Delta} = -\frac{\Delta}{\sqrt{\xi_k^2+\Delta^2}} = \frac{\Delta}{E_k}$$

Further, we note that $u_k v_k = \frac{1}{2}\sqrt{1 - \frac{\xi_k^2}{E_k^2}} = \frac{1}{2}\frac{\Delta}{E_k}$. Then

$$\begin{aligned}
\frac{\partial \Omega}{\partial \Delta} &= 0 \\
-2\sum_{\mathbf{k}} u_k v_k + 2\frac{L^3}{g}\Delta + 2\sum_{\mathbf{k}} \frac{\Delta/E_k}{1+e^{E_k/k_BT}} &= 0 \\
-2\sum_{\mathbf{k}} u_k v_k[1 - 2\frac{1}{1+e^{E_k/k_BT}}] + 2\frac{L^3}{g}\Delta &= 0
\end{aligned}$$

But this just equals the gap equation (158):

$$\Delta = \frac{g}{L^3}\sum_{\mathbf{k}} u_k v_k[1 - 2\frac{1}{e^{E_k/k_BT}+1}]$$

Done.