1. Show by using the GL differential equations that in GL theory the continuity equation $\nabla \cdot \mathbf{j} = 0$ and the boundary condition $\hat{\mathbf{n}} \cdot \mathbf{j} = 0$ are satisfied, where $\mathbf{j}$ is the current density and $\hat{\mathbf{n}}$ is the surface normal.

   Solution:
   The second GL equation is of the form

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{j}$$

   where the current density is given by

$$\mathbf{j} = \gamma\frac{\hbar}{i}q(\psi^*\nabla\psi - \psi\nabla\psi^*) - 2\gamma q^2|\psi|^2\mathbf{A}$$

   Now it follows quite trivially that

$$\nabla \cdot \mathbf{j} = \frac{1}{\mu_0}\nabla \cdot (\nabla \times \mathbf{B}) = 0$$

   Now let us rewrite the current density as

$$\mathbf{j} = \gamma q[\psi^*(\frac{\hbar}{i}\nabla - q\mathbf{A})\psi + \psi(-\frac{\hbar}{i}\nabla - q\mathbf{A})\psi^*]$$

   Now applying with the surface normal $\hat{\mathbf{n}}$ on this, we have

$$\hat{\mathbf{n}} \cdot \mathbf{j} = \gamma q[\psi^*\hat{\mathbf{n}} \cdot (\frac{\hbar}{i}\nabla - q\mathbf{A})\psi + \psi\hat{\mathbf{n}} \cdot (-\frac{\hbar}{i}\nabla - q\mathbf{A})\psi^*] = 0$$

   Here we applied $\hat{\mathbf{n}} \cdot (\frac{\hbar}{i}\nabla - q\mathbf{A})\psi = 0$, which was assumed as a boundary condition for the first GL equation. Thus that condition is equivalent requiring that no current passes through the boundary.

2. In the lecture notes the behavior of the magnetic field was derived in the case of a superconducting half space, when an external field is applied parallel to the surface. Calculate the related vector potential, current density and the total current. (Hint: If $\mathbf{B} = B(x)\hat{\mathbf{z}}$, you can assume that $\mathbf{A} = A(x)\hat{\mathbf{y}}$.)

   Solution:
   The exercise refers to the example 3 at page 25. The normal state occupies $x < 0$ and the superconductor $x > 0$. The planar interface is thus in the yz plane at $x = 0$.

The magnetic field is known to be of the form

$$\mathbf{B} = B_0 e^{-x/\lambda}\hat{\mathbf{z}}$$

for $x > 0$. The is related to the vector potential $\mathbf{A}$ by $\mathbf{B} = \nabla \times \mathbf{A}$, and we would like to find it. Assuming that $\mathbf{A} = A\hat{\mathbf{y}}$, we have

$$\mathbf{B} = \nabla \times (A\hat{\mathbf{y}}) = -\hat{\mathbf{x}}\frac{\partial A}{\partial z} + \hat{\mathbf{z}}\frac{\partial A}{\partial x}$$

Equating this with the above known epression of $\mathbf{B}$, we have

$$\frac{\partial A}{\partial x} = B_0 e^{-x/\lambda}$$

This can be integrated to

$$A(x) = -\lambda B_0 e^{-x/\lambda} + C$$

The constant C will be chosen so that the current density vanishes at $x \to \infty$, which gives $C = 0$, as shown below.

The simplest way to get the current density is to use the Maxwell equation:

$$\begin{aligned}\mathbf{j} &= \frac{1}{\mu_0}\nabla \times \mathbf{B} = \frac{1}{\mu_0}\nabla \times (B_0 e^{-x/\lambda}\hat{\mathbf{z}}) \\ &= \frac{B_0}{\mu_0}[-\hat{\mathbf{y}}\frac{\partial}{\partial x}(e^{-x/\lambda})] = \frac{B_0}{\mu_0\lambda}e^{-x/\lambda}\hat{\mathbf{y}}\end{aligned}$$

However, using the vector potential derived above, we get it also as follows. When the phase of the order parameter is assumed constant, the current density is

$$\mathbf{j} = -2q^2\gamma|\psi|^2\mathbf{A} = -\frac{1}{\mu_0\lambda^2}\mathbf{A} = \frac{1}{\mu_0\lambda^2}(\lambda B_0 e^{-x/\lambda} - C)\hat{\mathbf{y}} = \frac{B_0}{\mu_0\lambda}e^{-x/\lambda}\hat{\mathbf{y}}$$

where we used $|\psi|^2 = -\alpha/\beta = |\alpha|/\beta$ and $\lambda = \sqrt{\beta/(2q^2\mu_0\gamma|\alpha|)}$ and chose $C = 0$ so that the current density would vanish at infinity.

Total current. Let us assume that the z-directional height of the sample is L_z. Then

$$I = L_z\int_0^\infty dx j_y(x) = \frac{B_0 L_z}{\mu_0\lambda}\int_0^\infty e^{-x/\lambda}dx = \frac{B_0 L_z}{\mu_0\lambda}|_0^\infty(-\lambda)e^{-x/\lambda} = \frac{B_0 L_z}{\mu_0}$$

Note: As always, there is some freedom in the choice of the vector potential $\mathbf{A}$. The GL free energy functional and hence the current density have been constructed so that they are invariant under *gauge transformations* of the form $\mathbf{A} \to \mathbf{A} + \frac{\hbar}{q}\nabla\chi$ and $\phi \to \phi + \chi$, where ϕ is the phase of the order parameter and χ is an arbitrary function. (Check this. Note also that the electric scalar potential is simultaneously transformed as $V \to V - \frac{\hbar}{q}\dot{\chi}$, as always in electromagnetism, but this is not needed in our time-independent equilibrium GL theory.) Thus we could choose the constant C above differently, but then ϕ should be position-dependent such that the current is still the one given by the Maxwell equation.

Note 2: Just in case it was unclear, this problem concerns a planar *insulator–superconductor* (I-S) interface. In this case the boundary condition $\hat{\mathbf{n}}\cdot(\frac{\hbar}{i}\nabla - q\mathbf{A})\psi = 0$ implies that the normal derivative of the order parameter ψ is zero even when a magnetic field is applied (because in fact $\hat{\mathbf{n}}\cdot\mathbf{A} = 0$). Thus the amplitude of the superconducting order parameter remains finite all the way to the surface. In the case of a *normal metal–superconductor* (N-S) interface, ψ is suppressed close to the interface on a length scale ξ_{GL}. Don't confuse these things.

3. Show that by choosing units of length, energy, order parameter, and magnetic field properly, and neglecting constant energy terms, the GL free energy functional in a given external field $\mathbf{B}_{\text{ext}} = \nabla \times \mathbf{A}_{\text{ext}}$ at $T < T_c$ can be written in the dimensionless form

$$G(\Psi, \mathbf{A}) = \int d^3x \left[-|\Psi|^2 + \frac{1}{2}|\Psi|^4 + |(\nabla + \mathrm{i}\mathbf{A})\Psi|^2 + \kappa^2 |\nabla \times (\mathbf{A} - \mathbf{A}_{\text{ext}})|^2 \right],$$

which contains only one dimensionless parameter $\kappa = \lambda(T)/\xi_{GL}(T)$.

Solution:
The GL free energy is

$$G = \int d^3r [\alpha\psi^*\psi + \frac{\beta}{2}(\psi^*)^2\psi^2 + \gamma|(\frac{\hbar}{i}\nabla - q\mathbf{A})\psi|^2 + \frac{1}{2\mu_0}\mathbf{B}^2 - \mathbf{B}\cdot\mathbf{H}]$$

Let us modify the last two terms a bit.

$$\frac{1}{2\mu_0}\mathbf{B}^2 - \mathbf{B}\cdot\mathbf{H} = \frac{1}{2\mu_0}(\mathbf{B}^2 - \mu_0\mathbf{B}\cdot\mathbf{H}) = \frac{1}{2\mu_0}(\mathbf{B}^2 - 2\mu_0\mathbf{B}\cdot\mathbf{H} + \mu_0^2\mathbf{H}^2) - \frac{\mu_0}{2}\mathbf{H}^2$$
$$= \frac{1}{2\mu_0}(\mathbf{B} - \mu_0\mathbf{H})^2 - \frac{\mu_0}{2}\mathbf{H}^2 = \frac{1}{2\mu_0}(\mathbf{B} - \mathbf{B}_{ext})^2 - \frac{\mu_0}{2}\mathbf{H}^2$$

Here the last term is just a constant and may be dropped. Writing $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{B}_{ext} = \nabla \times \mathbf{A}_{ext}$ we can put the GL free energy in the form

$$G = \int d^3r \{\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \gamma|(\hbar\nabla - iq\mathbf{A})\psi|^2 + \frac{1}{2\mu_0}[\nabla \times (\mathbf{A} - \mathbf{A}_{ext})]^2\}$$

Now we define new dimensionless quantities and denote them with all with a tilde symbol. We do not yet know what are the "natural units" for energy and length, for example, but let us denote them with E_0 and L_0, respectively. So let us write

$$G = \tilde{G}E_0, \quad \mathbf{r} = \tilde{\mathbf{r}}L_0, \quad \nabla = \tilde{\nabla}/L_0, \quad \mathbf{A} = \tilde{\mathbf{A}}a, \quad \psi = \tilde{\psi}b$$

Here a and b are also unknown.

Now let's insert these into the free energy (we also write $\alpha = -|\alpha|$, since for $T < T_c$ it is true that $\alpha < 0$):

$$E_0\tilde{G} = \int d^3\tilde{r} L_0^3 \{-|\alpha|b^2|\tilde{\psi}|^2 + \frac{\beta}{2}b^4|\tilde{\psi}|^4 + \gamma b^2 \frac{\hbar^2}{L_0^2}|(\tilde{\nabla} - i\frac{qaL_0}{\hbar}\tilde{\mathbf{A}})\tilde{\psi}|^2 + \frac{1}{2\mu_0}\frac{a^2}{L_0^2}[\nabla \times (\mathbf{A} - \mathbf{A}_{ext})]^2\}$$

$$\tilde{G} = \int d^3\tilde{r} \{-|\alpha|\frac{b^2 L_0^3}{E_0}|\tilde{\psi}|^2 + \frac{\beta}{2}\frac{b^4 L_0^3}{E_0}|\tilde{\psi}|^4 + \gamma b^2 \frac{\hbar^2 L_0}{E_0}|(\tilde{\nabla} + i\tilde{\mathbf{A}})\tilde{\psi}|^2 + \frac{1}{2\mu_0}\frac{\hbar^2}{L_0 E_0 q^2}[\nabla \times (\mathbf{A} - \mathbf{A}_{ext})]^2\}$$

Here in the second form we aready required $\frac{qaL_0}{\hbar} = -1$, which gives us the first unknown as $a = -\hbar/(L_0 q) = \hbar/(L_0|q|)$, where we assumed $q < 0$. In order for this to be of the desired form, we must additionally have

$$|\alpha|\frac{b^2 L_0^3}{E_0} = 1, \quad \beta\frac{b^4 L_0^3}{E_0} = 1, \quad \gamma b^2\frac{\hbar^2 L_0}{E_0} = 1$$

From these three equations we must solve for the three unknowns L_0, E_0, and b. Whatever parameter combination remains in front of the fourth term in $\tilde{G}$ is then called κ^2. By dividing the first two of the three equations we find

$$\frac{|\alpha|}{\beta}\frac{1}{b^2} = 1 \qquad \Longrightarrow b = \sqrt{\frac{|\alpha|}{\beta}}$$

By inserting this b into the equations, we see that two independent equations remain for the two unknowns L_0, E_0:

$$\frac{|\alpha|^2}{\beta}\frac{L_0^3}{E_0} = 1, \quad \frac{|\alpha|\gamma}{\beta}\frac{\hbar^2 L_0}{E_0} = 1$$

Again, by dividing the two, we find

$$\frac{|\alpha|}{\gamma}\frac{L_0^2}{\hbar^2} = 1 \qquad \Longrightarrow L_0 = \sqrt{\frac{\gamma\hbar^2}{|\alpha|}} \equiv \xi_{GL}$$

This length scale ξ_{GL} is known as the GL coherence (or healing) length. Inserting this to one the two equations, we finally have

$$\frac{(|\alpha|)^{1/2}(\gamma)^{3/2}}{\beta}\frac{\hbar^3}{E_0} = 1 \qquad \Longrightarrow E_0 = \frac{|\alpha|^{1/2}\gamma^{3/2}\hbar^3}{\beta}$$

All of our former unknows are now known. Let us insert them into prefactor in front of the fourth term in $\tilde{G}$:

$$\frac{1}{2\mu_0}\frac{\hbar^2}{L_0 E_0 q^2} = \frac{1}{2\mu_0}\frac{\hbar^2}{q^2}\sqrt{\frac{|\alpha|}{\gamma\hbar^2}}\frac{\beta}{|\alpha|^{1/2}\gamma^{3/2}\hbar^3} = \frac{1}{2\mu_0}\frac{\hbar^2}{q^2} = \frac{1}{2\mu_0}\frac{\beta}{\gamma^2\hbar^2 q^2}$$

We want to isolate from this the proportionality to $L_0^{-2} = \xi_{GL}^{-2}$ coming from the ∇. So we continue

$$\frac{1}{2\mu_0}\frac{\hbar^2}{L_0 E_0 q^2} = \frac{1}{2\mu_0}\frac{\beta}{\gamma^2\hbar^2 q^2}\frac{\gamma\hbar^2}{|\alpha|}\frac{1}{\xi_{GL}^2} = \frac{1}{2\mu_0}\frac{\beta}{\gamma q^2|\alpha|}\frac{1}{\xi_{GL}^2} \equiv \frac{\lambda^2}{\xi_{GL}^2} \equiv \kappa^2$$

Here we defined the magnetic penetration length

$$\lambda = \sqrt{\frac{1}{2\mu_0}\frac{\beta}{\gamma q^2|\alpha|}}$$

and denoted the dimensionless ratio λ/ξ_{GL} by κ. In this way we finally have

$$\tilde{G}[\tilde{\psi}, \tilde{\mathbf{A}}] = \int d^3\tilde{r}\{-|\tilde{\psi}|^2 + \frac{1}{2}|\tilde{\psi}|^4 + |(\tilde{\nabla} + i\tilde{\mathbf{A}})\tilde{\psi}|^2 + \kappa^2[\nabla \times (\tilde{\mathbf{A}} - \tilde{\mathbf{A}}_{ext})]^2\}$$

where only some symbols differ from the desired form.

4. Consider the one-dimensonal GL equation

$$\xi_{GL}^2 \frac{d^2 f}{dx^2} + f - f^3 = 0.$$

Its first integral can be derived by analogy, by comparing the GL energy to the action integral $S = \int dt L(\{\dot{q}_i\}, \{q_i\}, t)$ where $L = T - V$, and noting that when $\partial L/\partial t = 0$, the Hamiltonian $H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$ is constant due to the equation of motion $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$ (which in this analogy is the GL equation). Do this. Show that

$$f(x) = \tanh \frac{x}{\sqrt{2}\xi_{GL}}$$

satisfies the first-integral equation. You can find the first integral also without the analogy and solve it for f directly, if you prefer that.

Solution:
The one-dimensional GL energy functional in the absence of a magnetic field is

$$G = \int dx(\alpha|\psi|^2 + \frac{1}{2}|\psi|^4 + \gamma\hbar^2|\frac{d\psi}{dx}|^2)$$

We assume ψ to be real and write $\psi = \sqrt{|\alpha|/\beta} f$:

$$\begin{aligned}
G &= \int dx[-\frac{|\alpha|^2}{\beta}f^2 + \frac{1}{2}\frac{|\alpha|^2}{\beta}f^4 + \gamma\hbar^2\frac{|\alpha|}{\beta}(\frac{d\psi}{dx})^2] \\
&= \frac{|\alpha|^2}{\beta}\int dx[-f^2 + \frac{1}{2}f^4 + \frac{\gamma\hbar^2}{|\alpha|}(\frac{d\psi}{dx})^2] \\
&= \frac{|\alpha|^2}{\beta}\int dx[-f^2 + \frac{1}{2}f^4 + \xi_{GL}^2(\frac{df}{dx})^2]
\end{aligned}$$

Or defining $\bar{G} = G\beta/|\alpha|^2$

$$\bar{G} = \int dx[-f^2 + \frac{1}{2}f^4 + \xi_{GL}^2(\frac{df}{dx})^2]$$

(Note: this $\bar{G}$ is not dimensionless.) In analytical mechanics one defines the action integral $S = \int dtL(\dot{q}, q, t)$ where $L = T - V$, the difference of the kinetic and potential energies. The above GL integral is clearly of this form, just with the replacement of t by x, and so we can rely on some analogies. Thus we define

$$\bar{G} = \int dxL(f', f, x),$$

where the "Lagrangian" is

$$L(f', f, x) = \underbrace{\xi_{GL}^2(f')^2}_{T} - \underbrace{(f^2 - \frac{1}{2}f^4)}_{V}$$

and where prime denotes x-derivative. There is no explicit x dependence, $\partial L/\partial x = 0$, so that the Lagrangian equation of motion is just $\frac{d}{dx}\frac{\partial L}{\partial f'} - \frac{\partial L}{\partial f} = 0$, which gives the 1-D GL equation

$$\xi_{GL}^2 f'' + f - f^3 = 0$$

The fact that $\partial L/\partial x = 0$ means that the corresponding "Hamiltonian" $H =$ constant:

$$\begin{aligned} H &= f'\frac{\partial L}{\partial f'} - L = f'(2\xi_{GL}^2 f') - [\xi_{GL}^2(f')^2 - (f^2 - \frac{1}{2}f^4)] \\ &= \xi_{GL}^2(f')^2 + f^2 - \frac{1}{2}f^4 \end{aligned}$$

This is thus the "first integral" of the 1-D GL equation. You can check that by differentiating this on both sides gives the GL equation.

The first integral can be solved by the given Ansatz $f(x) = \tanh(x/\sqrt{2}\xi_{GL})$, with

$$f'(x) = (1 - \tanh^2 \frac{x}{\sqrt{2}\xi_{GL}})\frac{1}{\sqrt{2}\xi_{GL}}$$

Inserting these we have

$$\begin{aligned} H &= \xi_{GL}^2[(1 - \tanh^2 \frac{x}{\sqrt{2}\xi_{GL}})\frac{1}{\sqrt{2}\xi_{GL}}]^2 + \tanh^2 \frac{x}{\sqrt{2}\xi_{GL}} - \frac{1}{2}\tanh^2 \frac{x}{\sqrt{2}\xi_{GL}} \\ &= \frac{1}{2} - \tanh^2 \frac{x}{\sqrt{2}\xi_{GL}} + \frac{1}{2}\tanh^4 \frac{x}{\sqrt{2}\xi_{GL}} + \tanh^2 \frac{x}{\sqrt{2}\xi_{GL}} - \frac{1}{2}\tanh^4 \frac{x}{\sqrt{2}\xi_{GL}} \\ &= \frac{1}{2} \end{aligned}$$

So the Hamiltonian, or the first integral, is indeed constant for this form of $f(x)$. Thus $f(x) = \tanh(x/\sqrt{2}\xi_{GL})$ is a solution to the one-dimensional GL equation.

We could solve the first integral also directly. First of all, the GL equation gives (let $\xi_{GL} \to \xi$)

$$\begin{aligned}
-\xi^2 f'' - f + f^3 &= 0 \\
-\xi^2 f'' f' - f f' + f^3 f' &= 0 \\
\frac{d}{dx}[\frac{1}{2}\xi^2 (f')^2 + \frac{1}{2} f^2 - \frac{1}{4} f^4] &= 0 \\
\frac{d}{dx}[\xi^2 (f')^2 + f^2 - \frac{1}{2} f^4] &= 0 \\
\xi^2 (f')^2 + f^2 - \frac{1}{2} f^4 &= H
\end{aligned}$$

where H is a constant. This is just as above. Now to have $f(x \to \infty) = 1$, we must have $H = 1/2$. So

$$\begin{aligned}
\xi^2 (f')^2 - \frac{1}{2}(1 - 2f^2 + f^4) &= 0 \\
\xi^2 (f')^2 - \frac{1}{2}(1 - f^2)^2 &= 0
\end{aligned}$$

Choose $f' > 0$. Then

$$f' = \frac{1}{\sqrt{2}\xi}(1 - f^2)$$

Separate variables:

$$\begin{aligned}
&\int \frac{df}{1 - f^2} = \frac{1}{\sqrt{2}\xi} \int dx \quad \to \\
&\text{artanh} f = (x - x_0)/\sqrt{2}\xi \quad \to \quad f(x) = \tanh[(x - x_0)/\sqrt{2}\xi]
\end{aligned}$$

Here x_0 is a constant. If $f(0) = 0$, then $x_0 = 0$.

5. Calculate the discontinuity ΔC of the specific heat at $T = T_c$ in the G-L theory. Using the microscopic (BCS) values for the G-L parameters and the known result for the specific heat $C_n(T)$ of the normal state, show that

$$\frac{\Delta C}{C_n(T_c)} = \frac{12}{7\zeta(3)} = 1.43.$$

(Hint: The specific heat is, as usual $C = -T(\partial^2 G/\partial T^2)$.)

Solution:
We have previously calculated the specific heat for the normal state:

$$C_n(T) = \frac{2\pi^2}{3} L^3 N(0) k_B^2 T$$

The specific heat for the superconducting state can be calculated from the free energy G as

$$C_s(T) = -T \left(\frac{\partial^2 G}{\partial T^2} \right)_{H,V,\mu}$$

where the free energy is calculated in the G-L theory from the functional

$$G = F_0 + \int d^3r [\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \gamma|(\frac{\hbar}{i}\nabla - q\mathbf{A})\psi|^2 + \frac{1}{2\mu_0}\mathbf{B}^2 - \mathbf{B}\cdot\mathbf{H}]$$

where F_0 is the free energy of normal state. At $T = T_c$ there cannot be magnetic fields present. We also assume the superconducting state to be spatially homogeneous, meaning that $\mathbf{A} = \mathbf{B} = 0$ and $\nabla\psi = 0$ and then $\psi = \sqrt{\frac{|\alpha|}{\beta}}$, as suggested by Eq. (218). The G-L energy thus reduces to

$$G = F_0 + \int d^3r \left(-\frac{|\alpha|^2}{\beta} + \frac{1}{2}\frac{|\alpha|^2}{\beta} \right) = F_0 - \frac{|\alpha|^2 L^3}{2\beta}$$

For this calculation we need the microscopic values for the G-L parameters, which are given in the lecture notes. They follow from BCS theory:

$$\alpha = N(0)\frac{T - T_c}{T_c}$$

$$\beta = \frac{7\zeta(3)N(0)}{8\pi^2 k_B^2 T_c^2}$$

By substituting these into the energy:

$$G = F_0 - \frac{4N(0)\pi^2 k_B^2 L^3 (T - T_c)^2}{7\zeta(3)}$$

Now

$$C_s(T) = -T \left(\frac{\partial^2 G}{\partial T^2} \right)_{H,V,\mu} = C_n(T) + \frac{8N(0)\pi^2 k_B^2 L^3 T}{7\zeta(3)}$$

where $C_n(T) = -T\frac{\partial^2 F_0}{\partial T^2}$ has the expression quoted above. Then

$$\frac{\Delta C(T_c)}{C_n(T_c)} = \frac{C_s(T_c) - C_n(T_c)}{C_n(T_c)} = \frac{\frac{8N(0)\pi^2 k_B^2 L^3 T_c}{7\zeta(3)}}{\frac{2\pi^2}{3} L^3 N(0) k_B^2 T_c}$$

$$= \frac{12}{7\zeta(3)} = \frac{12}{7 \cdot 1.203} = 1.43$$