1. Consider G-L equations in a thin wire, assuming that $\mathbf{A} = 0$ and

$$\Psi(x) = Ce^{ikx}.$$

Calculate the supercurrent j and the G-L energy for this state. Minimize the energy with respect to C at constant k. Describe C and j as functions of k. Find the maximum supercurrent and the corresponding k.

Solution:
As instructed, we consider a solution to the 1-D G-L equations of the form $\psi(x) = Ce^{ikx}$. First the supercurrrent:

$$\begin{aligned} j &= \frac{q\hbar\gamma}{i}(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}) \\ &= 2q\hbar\gamma|C|^2 k = 2q\hbar\gamma C^2 k \end{aligned}$$

In the last step we exercised our freedom to choose C to be real. Next we calculate the G-L free energy and minimize:

$$\begin{aligned} G &= \int dx[\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \gamma\hbar^2|\frac{d\psi}{dx}|^2] \\ G &= \int dx[-|\alpha|C^2 + \frac{\beta}{2}C^4 + \gamma\hbar^2 C^2 k^2] \\ G &= L[-|\alpha|C^2 + \frac{\beta}{2}C^4 + \gamma\hbar^2 C^2 k^2] \end{aligned}$$

Here we considered a system of length L. Minimization of the free energy with respect to C:

$$\begin{aligned} &\frac{\partial G}{\partial C} = 0, \quad L[2(-|\alpha| + \gamma\hbar^2 k^2)C + 2\beta C^3] = 0 \\ &C^2 = \frac{|\alpha| - \gamma\hbar^2 k^2}{\beta} \end{aligned}$$

(Note that this same equation would actually follow by inserting $\psi(x) = Ce^{ikx}$ directly to the 1-D G-L equation $\hbar^2\gamma\nabla^2\psi - \alpha\psi - \beta|\psi|^2\psi = 0$.) There is another solution $C = 0$, but that is not what we are interested in here. However, also the other solution vanishes if k reaches a value $k = k_{c2} = \sqrt{|\alpha|/\gamma\hbar^2} = 1/\xi_{GL}$. The system can be in the superconducting state only for $k < k_{c2}$, and then the amplitude of the order parameter is

$$C(k) = \sqrt{\frac{|\alpha| - \gamma\hbar^2 k^2}{\beta}} = \sqrt{\frac{\gamma\hbar^2}{\beta}(k_{c2}^2 - k^2)}$$

Now let us insert this into the supercurrent.

$$j(k) = 2q\hbar\gamma C^2 k = 2q\hbar\gamma k \frac{|\alpha| - \gamma\hbar^2 k^2}{\beta} = \frac{2q\hbar\gamma}{\beta}(|\alpha|k - \gamma\hbar^2 k^3)$$

The maximal supercurrent is found from $dj/dk = 0$, which gives $k = k_{c1} = \sqrt{|\alpha|/3\gamma\hbar^2} = 1/(\sqrt{3}\xi_{GL}) < k_{c2}$. The maximal value of the current is then

$$j_c = j(k_{c1}) = \frac{2q\hbar\gamma}{\beta}(|\alpha|\frac{1}{\sqrt{3}\xi_{GL}} - |\alpha|\frac{1}{3\sqrt{3}\xi_{GL}}) = \frac{q\hbar\gamma|\alpha|}{\beta\xi_{GL}}\frac{4}{3\sqrt{3}}$$

where we used $k_{c1}^3 = 1/(3\sqrt{3}\xi_{GL}^3)$ and $\gamma\hbar^3/|\alpha| = \xi_{GL}^2$. This is the "critical" current density, at least one way of writing it.

2. Calculate the energy of an interface between normal and superonducting states (in the critical field H_c) in the limit $\kappa \to 0$, in which case the you can neglect the magnetic field on the superconducting side and use the solution $f(x) = \tanh\left(x/\sqrt{2}\xi_{GL}\right)$. Note again that the free energy densities of the normal and superconducting states (at $x = \pm\infty$) have to be the same in order to have a stable interface.

   (Hint: In this limit the N-S interface is abrupt, and you can choose it to be at $x = 0$ for example. At this point f is continuous. On the N side $B = \mu_0 H_c$ and $f = 0$.)

   Solution:

We assume the normal metal (N) to occupy the space $x \lesssim 0$ and the superconductor (S) $x \gtrsim 0$. Normal and superconducting states in the same material can only coexist at the critical field, so we assume $H = H_c$. Hence we cannot simply neglect the magnetic field. However, in the limit $\kappa = \lambda/\xi_{GL} \to 0$ the magnetic field B cannot exist in the same region where $\psi \neq 0$. This simplifies the calculation because the N-S interface becomes clear-cut so that we can choose N ($\psi = 0$, $B \neq 0$) to occupy the region $x < 0$ and S ($B = 0$, $\psi \neq 0$) the region $x > 0$.

In equilibrium, the free energy densities of the normal state and the bulk of the superconductor are equal $g_s = f_0 - \frac{1}{2}\frac{|\alpha|^2}{\beta} = g_n = f_0 - \frac{1}{2}\mu_0 H_c^2$. Only in the interface region does the energy density $g(x)$ deviate from this value. Thus the "interface energy" (per area S of interface) is well defined when calculated with respect to a state where the energy density is equal to $g_s = g_n$ everywhere. (So either with respect to pure superconducting state or pure normal state.) Thus, it is defined as

$$\sigma = \Delta G/S = \int_{-\infty}^{\infty} dx[g(x) - g_{n,s}]$$

where

$$g(x) = f_0 + \frac{|\alpha|^2}{\beta}\Big\{ - f^2(x) + \frac{1}{2}f^4(x) + \xi_{GL}^2[f'(x)]^2 + \frac{q^2\gamma}{|\alpha|}A^2(x)f^2(x) \\ + \frac{\beta}{2\mu_0|\alpha|^2}([A'(x)]^2 - 2\mu_0 A'(x)H)\Big\}$$

where $A(x)$ is the y component of vector potential, assuming the magnetic field to point in the direction z. This energy density is derived in another exercise. Here it is not needed in this general form, because in the limit $\kappa \to 0$ ($\lambda \ll \xi_{GL}$) we can set $f = 0$ on the normal side and $A = 0$ on the superconducting side. Assuming the N-S interface to be at $x = 0$, in the normal region $x < 0$ we have $g(x) = g_n$ and the interface energy simplifies to

$$\sigma = \Delta G/S = \int_0^\infty dx[g(x) - g_s]$$

with

$$g(x) = f_0 + \frac{|\alpha|^2}{\beta}\Big\{ - f^2(x) + \frac{1}{2}f^4(x) + \xi_{GL}^2[f'(x)]^2\Big\}$$

We could have written down this form also directly. The integral is thus

$$\sigma = \frac{|\alpha|^2}{\beta}\int_0^\infty dx\Big\{ - f^2(x) + \frac{1}{2}f^4(x) + \xi_{GL}^2[f'(x)]^2 + \frac{1}{2}\Big\}$$

Here we may now insert $f(x) = \tanh(x/\sqrt{2}\xi_{GL})$, which has previously been shown to solve the 1-D G-L equation, and it therefore mimimizes the free energy. Note that $f(0) = 0$, so $f(x)$ is continuous across the interface as it should. (If the interface were at $x = x_0$, then we could simply shift the solution so that $f(x_0) = 0$.) So using

$$f'(x) = \frac{1}{\sqrt{2}\xi_{GL}}[1 - \tanh^2(\frac{x}{\sqrt{2}\xi_{GL}})] = \frac{1}{\sqrt{2}\xi_{GL}}(1 - f^2)$$

we have

$$\begin{aligned}
\sigma &= \frac{|\alpha|^2}{\beta}\int_0^\infty dx\Big\{ - f^2 + \frac{1}{2}f^4 + \frac{1}{2}(1 - f^2)^2 + \frac{1}{2}\Big\} \\
&= \frac{|\alpha|^2}{\beta}\int_0^\infty dx\Big\{1 - 2f^2 + f^4\Big\} \\
&= \frac{|\alpha|^2\sqrt{2}\xi_{GL}}{\beta}\int_0^\infty dx\Big\{(1 - f^2)\underbrace{\frac{1}{\sqrt{2}\xi_{GL}}(1 - f^2)}_{f'}\Big\} \\
&= \frac{|\alpha|^2\sqrt{2}\xi_{GL}}{\beta}\int_0^\infty dx(1 - f^2)f' = \frac{|\alpha|^2\sqrt{2}\xi_{GL}}{\beta}\Big|_0^\infty (f - \frac{1}{3}f^3) = \frac{2\sqrt{2}|\alpha|^2\xi_{GL}}{3\beta}
\end{aligned}$$

This is the interface energy. Note that it is > 0, which is the defining property of a type I superconductor. This is consistent with our assumption $\kappa \ll 1$.

Note: We did not utilize our knowledge of the "first integral" $\xi_{GL}^2(f')^2 + f^2 - \frac{1}{2}f^4 = \frac{1}{2}$ above. Somehow I have the feeling that that could have been useful. Perhaps you can spot how.

3. Determine the density of vortices (number per cross-sectional area) in a rotating superfuid by starting from the assumption that the velocity on the edge of the cylindrical container is on average the same as the velocity of the edge. How many vortices are there in a cylinder of radius 5 mm that makes one revolution per second? Consider separately ^{3}He and ^{4}He.

   (Hint: The circulation $\oint d\mathbf{l} \cdot \mathbf{v}_s$ around N vortices is $N\frac{h}{m}$.)

   Solution:
   The line integral of $\mathbf{v}_s$ satisfies

$$\oint d\mathbf{l} \cdot \mathbf{v}_s = N\frac{h}{m}$$

   where N is the number of vortices enclosed in the integration loop. This can also be formulated so that a vortex at (2D) location $\mathbf{r_i}$ creates a vorticity $\nabla \times \mathbf{v}_s = \hat{\mathbf{z}}\frac{h}{m}\delta(\mathbf{r} - \mathbf{r}_i)$. Thus if there are N vortices, $\nabla \times \mathbf{v}_s = \hat{\mathbf{z}}\frac{h}{m}\sum_{i=1}^{N}\delta(\mathbf{r} - \mathbf{r}_i)$, and by the Stokes theorem $\oint d\mathbf{l} \cdot \mathbf{v}_s = \int d\mathbf{a} \cdot \nabla \times \mathbf{v}_s = N\frac{h}{m}$.

   If the fluid would move completely with the container, then the velocity at any point $\mathbf{r}$ would be given by $\mathbf{v}_s = \mathbf{\Omega} \times \mathbf{r} = \Omega r\hat{\boldsymbol{\phi}}$ where $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$, with Ω the angular velocity of the container. (The vorticity would then be $\nabla \times \mathbf{v}_s = 2\mathbf{\Omega}$, as you may check.) The vortex-filled superfluid mimics this behavior "on average", because the vortex lattice rotates with the container.

   Let the radius of the container be R. Then the cross-sectional area is $A = \pi R^2$. The density of vortices is $n = N/A$. The average velocity on the edge of the container is $v_s = \Omega R$. Thus we may write

$$\begin{aligned} 2\pi R v_s &= An\frac{h}{m} \\ 2\pi\Omega R^2 &= \pi R^2 n\frac{h}{m} \\ 2\Omega &= n\frac{h}{m} \\ n &= \frac{2m\Omega}{h} \end{aligned}$$

   The connection between angular velocity Ω and the frequency f (or rotation period $T = 1/f$) is $\Omega = 2\pi f$, so we may write this also as

$$n = \frac{2m(2\pi f)}{h} = \frac{2mf}{(h/2\pi)} = \frac{2mf}{\hbar}$$

Now consider a cylinder with radius $R = 5$ mm, and $f = 1$ Hz. For ^{3}He the mass is $m = 2m_3 = 6u$ and for For ^{4}He $m = m_4 = 4u$, where u is the atomic mass unit:

$$N_3 = An_3 = \pi R^2 \frac{2(2m_3)f}{\hbar} = \pi \cdot (5 \cdot 10^{-3}\ \text{m})^2 \cdot \frac{2 \cdot 6 \cdot 1.6605 \cdot 10^{-27}\ \text{kg} \cdot 1\ \text{Hz}}{1.0546 \cdot 10^{-34}\ \text{Js}} \approx 14840$$
$$N_4 = An_4 = \pi R^2 \frac{2m_4 f}{\hbar} = \pi \cdot (5 \cdot 10^{-3}\ \text{m})^2 \cdot \frac{2 \cdot 4 \cdot 1.6605 \cdot 10^{-27}\ \text{kg} \cdot 1\ \text{Hz}}{1.0546 \cdot 10^{-34}\ \text{Js}} \approx 9894$$

Quite many, it seems.

Note: The fact that the vortex lattice rotates together with the container, approximating solid-body rotation, follows from minimizing the free energy in a coordinate system rotating with the container. (You can only have equilibrium in a reference frame where external potentials are time-independent.) The energy to be minimized should be of the form $F = F_{\Omega=0} - \mathbf{\Omega} \cdot \mathbf{L}$ where $\mathbf{L}$ is the total angular momentum $\mathbf{L} = \int d^3r \mathbf{r} \times (\rho_s \mathbf{v}_s)$. Apart from constants, this gives $F = \int d^3r \frac{1}{2}\rho_s(\mathbf{v}_s - \mathbf{\Omega} \times \mathbf{r})^2$, where $\mathbf{v}_s = \frac{\hbar}{m}\nabla\phi$. Here $\mathbf{v}_s$ is the velocity in the inertial frame, and $\mathbf{v}_s - \mathbf{\Omega} \times \mathbf{r}$ the velocity in the rotating frame. So in the vortex configuration and associated velocity field that minimized the energy, averaging over a few sites in the vortex lattice we should have $\langle \mathbf{v}_s \rangle_{ave} = \mathbf{\Omega} \times \mathbf{r}$, although this cannot be satisfied exactly for $\mathbf{v}_s$ at every point in space.

4. In a superconductor one can define the velocity of the superconducting part as

$$\mathbf{v}_s = \frac{1}{m}(\hbar\nabla\chi - q\mathbf{A}).$$

When rotating the superconductor, no vortices are generated, but a uniform magnetic field is. Calculate it. You can assume the condensate to rotate as a solid body together with the atomic lattice.

Solution:

The superfluid velocity is

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla\chi - \frac{q}{m}\mathbf{A}$$

Consider some closed integration path within the superconductor. Then

$$\oint d\mathbf{l} \cdot \mathbf{v}_s = \frac{\hbar}{m}\oint d\mathbf{l} \cdot \nabla\chi - \frac{q}{m}\oint d\mathbf{l} \cdot \mathbf{A}$$

Assuming that there are no vortices, the first term on the right-hand side vanishes: $\oint d\mathbf{l} \cdot \nabla\chi = 0$. Then by using the Stokes formula

$$\oint d\mathbf{l} \cdot \mathbf{v}_s = -\frac{q}{m}\oint d\mathbf{l} \cdot \mathbf{A}$$
$$\int d\mathbf{S} \cdot \nabla \times \mathbf{v}_s = -\frac{q}{m}\int d\mathbf{S} \cdot \nabla \times \mathbf{A} = -\frac{q}{m}\int d\mathbf{S} \cdot \mathbf{B}$$

In vector form we can similarly conclude that $\nabla \times \nabla \chi = 0$ and so

$$\nabla \times \mathbf{v}_s = -\frac{q}{m} \nabla \times \mathbf{A} = -\frac{q}{m} \mathbf{B}$$

For an uncharged superfluid with $q = 0$ this (absence of vortices) would imply $\nabla \times \mathbf{v}_s = 0$.

Now consider a superconductor being rotated at an angular velocity $\mathbf{\Omega} = \Omega \hat{\mathbf{z}}$ around some orbitrary origin $\mathbf{r} = 0$. If the superfluid were uncharged, linear vortices would form, in the cores of which the "vorticity" $\nabla \times \mathbf{v}_s$ is locally nonzero. The vortex lattice would then mimic the solid-body rotation on average. However, in the case of a charged superfluid, this is not the only possibility and, indeed, does not happen (see note below). Instead of forming vortices, the superconducting electrons can rotate together with the lattice like a solid body, having the velocity

$$\mathbf{v}_s = \mathbf{\Omega} \times \mathbf{r}$$

and thus the vorticity

$$\nabla \times \mathbf{v}_s = 2\mathbf{\Omega}$$

which follows by applying $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a})$ and noting that $\nabla \cdot \mathbf{r} = 3$ etc. So, equating now the two forms $\nabla \times \mathbf{v}_s = -\frac{q}{m}\mathbf{B}$ and $\nabla \times \mathbf{v}_s = 2\mathbf{\Omega}$ we find that

$$\mathbf{B} = -\frac{2m}{q} \mathbf{\Omega}$$

Thus a magnetic field proportional to the angular velocity is generated. This magnetic field has been measured, and has precisely the magnitude given by the above formula — see for example A. F. Hildebrandt, Phys. Rev. Lett. **12**, 190 (1964). Note that here the mass m and the charge q must both be those of the "Cooper pair", or then of a single electron.

Note: To see that the condensate chooses to rotate with the lattice and form a uniform magnetic field instead of generating vortices, one should look at the energy functional in a rotating coordinate system and minimize that. The energy to be minimized is of the form $F = \frac{1}{2} \int d^3 r \rho_s (\mathbf{v}_s - \mathbf{\Omega} \times \mathbf{r})^2 + \frac{B^2}{2\mu_0}$. (Probably you should assume $\kappa \gg 1$?) You cannot minimize both the first and second terms at the same time, and so it is a matter of which one "costs more" to leave unminimized. Apparently it is usually favorable to minimize the first term so that $\mathbf{v}_s = \mathbf{\Omega} \times \mathbf{r}$, although the magnetic field energy $B^2/2\mu_0 \propto \Omega^2$ is then large. In the case of an uncharged superfluid, the only option would be to form vortices. Then you can still have $\langle \mathbf{v}_s \rangle_{ave} = \mathbf{\Omega} \times \mathbf{r}$.