1. Verify that the dimension of length is in accordance with the given field of 1 Tesla in the experimental vortex lattice picture given in the lecture notes.

   Solution:
   This concerns the figure on p. 28 of the 2015 lecture notes. The number of vortex lines is 42. So the total flux is $\Phi = 42\Phi_0 = 42(h/2e) = 8.7 \cdot 10^{-14}$ Wb. The area is roughly $A = 300 \times 300\ \mathrm{nm}^2 = 9 \cdot 10^{-14}\ \mathrm{m}^2$, so the expected flux at $B = 1$ T is $\Phi = BA = 9 \cdot 10^{-14}$ Wb. Seems to be right.

2. Show that the density of the single-particle energies $E_k = \sqrt{\xi_k^2 + \Delta^2}$ (i.e. density of states) in a superconductor is 0 for $0 < E < \Delta$ and

$$N_s(E) = \frac{N_n(0)E}{\sqrt{E^2 - \Delta^2}}$$

   for $E > \Delta$, where $N_n(0)$ is the corresponding normal-state ($\Delta = 0$) density of states.

   Solution:
   In the following we adopt a "particle picture" rather than "excitation picture", so that the energy dispersion is

$$E_k = \begin{cases} \sqrt{\xi_k^2 + \Delta^2}, & \xi_k > 0 \\ -\sqrt{\xi_k^2 + \Delta^2}, & \xi_k < 0 \end{cases}$$

   where $\xi_k = \epsilon_k - \mu$. Thus the "hole" states, defined here by $\xi_k < 0$, now appear at negative energies. (This is done just to make ξ_k uniquely defined for given E_k, which makes the discussion a bit cleaner — see below. If it bothers you, consider only the case $\xi_k > 0$.) In the normal state ($\Delta = 0$) this dispersion reduces to $E_k = \xi_k$. Since the density of the states on the "ξ axis" is therefore the same for both normal and superconducting cases, no states are "lost" in the superconducting transition. They just get redistributed on the "E axis".

   Therefore the normal-state DOS $N_n(\xi)$ and the superconducting-state DOS $N_s(E)$ should satisfy (for $E > \Delta$, say) "$N_s(E_s)dE_s = N_n(E_n)dE_n$", that is

$$N_s(E)dE = N_n(\xi)d\xi$$

   where $E = \sqrt{\xi^2 + \Delta^2}$ or $\xi = \sqrt{E^2 - \Delta^2}$. Thus by differentiation we find $d\xi = EdE/\sqrt{E^2 - \Delta^2}$ and so

$$N_s(E) = N_n(0)\frac{E}{\sqrt{E^2 - \Delta^2}}, \quad E > \Delta$$

where we additionally assumed that $N_n(\xi) \approx N_n(0)$. For $0 < E < \Delta$ we obviously have $N_s(E) = 0$. For $E < 0$ similar arguments can be given, so that in general

$$N_s(E) = N_n(0)\frac{|E|}{\sqrt{E^2 - \Delta^2}}, \quad |E| > \Delta$$

and $N_s(E) = 0$ for $|E| < \Delta$. In the "excitation picture" also the "hole" states would be counted at positive energies, and then you have to think of including a factor 2 in $N_n(0)$ in the above result and only consider $E > 0$. So then $N_n(0) = 2N(0)$, where $N(0)$ is our usual notation for the single-spin DOS around Fermi energy.

A more technical way is roughly as follows. The energy density of states per unit volume is defined as

$$N_s(E) = \frac{1}{L^3}\sum_{\mathbf{k}} \delta(E - E_k)$$

where $E_k = \text{sign}(\xi_k)\sqrt{\xi_k^2 + \Delta^2}$. Since $|E_k|$ can only have values $> \Delta$, it is clear that $N_s(E) = 0$ if $|E| < \Delta$. So below we assume $|E| > \Delta$. Here we need a delta function formula. In general, for a function $g(x)$ with zeros at $x = x_i$ (that is $g(x_i) = 0$) we have

$$\delta[g(x)] = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

Applying this to our case, we define

$$g(\xi_k) = E - \text{sign}(\xi_k)\sqrt{\xi_k^2 + \Delta^2}$$

This has zeros at $\xi_k = \xi_k^{(0)}$ with $\xi_k^{(0)} = \text{sign}(E)\sqrt{E^2 - \Delta^2}$. Now

$$g'(\xi_k) = \frac{|\xi_k|}{\sqrt{\xi_k^2 + \Delta^2}}$$

and

$$g'(\xi_k^{(0)}) = \frac{\sqrt{E^2 - \Delta^2}}{|E|}$$

Thus

$$N_s(E) = \frac{1}{L^3}\sum_{\mathbf{k}} \frac{|E|}{\sqrt{E^2 - \Delta^2}}\delta(\xi_k - \xi_k^{(0)})$$

Then we use the usual substitution of the sum by an integral, assuming the normal-state density of states to be constant

$$N_s(E) = N_n(0)\int d\xi_k \frac{|E|}{\sqrt{E^2 - \Delta^2}}\delta(\xi_k - \text{sign}(E)\sqrt{E^2 - \Delta^2})$$

The integral over the delta function gives the result 1 and so

$$N_s(E) = N_n(0)\frac{|E|}{\sqrt{E^2 - \Delta^2}}, \quad |E| > \Delta$$

For any energy E we may write

$$N_s(E) = N_n(0)\frac{|E|}{\sqrt{E^2 - \Delta^2}}\theta(|E| - \Delta)$$

where $\theta(x)$ is the Heaviside step function.

3. Show, as instructed in the lecture notes, that if the Jospehson coupling energy is $F_J(\Delta\phi) = -E_J \cos\Delta\phi$, then the Josephson current is $J = \frac{|q|}{\hbar}E_J \sin\Delta\phi$. Thus consider for simplicity a quasi-one-dimensional model, with a first superconductor at $-L < x < 0$ connected to a second one at $0 < x < L$ via a tunnel barrier at $x = 0$. The energy is of the form $F = S\int_{-L}^{0} dx f(x) + S\int_0^L dx f(x) + F_J$, where S is a cross-sectional area, $f(x)$ is the GL energy density at zero magnetic field, with $\psi(x) = \psi_0 e^{i\phi(x)}$, where ψ_0 is a real constant, and $\Delta\phi = \phi(0^-) - \phi(0^+)$. By considering variations $\phi(x) \to \phi(x) + \delta\phi(x)$ where $\delta\phi(\pm L) = 0$, show that the equilibrium conditions arising from the surface terms of $\delta F = 0$ at $x = 0^\pm$ imply that the current $J = Sj(0^-) = Sj(0^+)$ satisfies $J = \frac{|q|}{\hbar}\partial F_J(\Delta\phi)/\partial\Delta\phi$.

Solution:
Denote the phase fields at $x < 0$ and $x > 0$ by ϕ_1 and ϕ_2, respectively. Since the amplitudes $|\psi| = \psi_0$ of the order parameters are just constants, GL free energy is then of the form

$$F[\phi_1, \phi_2] = S\int_{-L}^{0} dx[f + \hbar^2\gamma|\psi|^2(\phi_1')^2] + S\int_0^L dx[f + \hbar^2\gamma|\psi|^2(\phi_2')^2] + F_J[\phi_1(0^-) - \phi_2(0^+)]$$

where $f = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$ is an unimportant constant energy density and S is the cross-sectional area. By differentiation

$$\begin{aligned}\delta F = {} & S\int_{-L}^{0} dx[2\hbar^2\gamma|\psi|^2\phi_1'(\delta\phi_1)'] + S\int_0^L dx[2\hbar^2\gamma|\psi|^2\phi_2'(\delta\phi_2)'] \\ & + F_J'[\phi_1(0^-) - \phi_2(0^+)][\delta\phi_1(0^-) - \delta\phi_2(0^+)]\end{aligned}$$

and by integration by parts

$$\begin{aligned}
\delta F = {} & S\Big|_{-L}^{0}(2\hbar^2\gamma|\psi|^2\phi_1')\delta\phi_1 - S\int_{-L}^{0} dx(2\hbar^2\gamma|\psi|^2\phi_1'')\delta\phi_1 \\
& + S\Big|_{0}^{L}(2\hbar^2\gamma|\psi|^2\phi_2')\delta\phi_2 - S\int_{0}^{L} dx(2\hbar^2\gamma|\psi|^2\phi_2'')\delta\phi_2 \\
& + F_J'[\phi_1(0^-) - \phi_2(0^+)][\delta\phi_1(0^-) - \delta\phi_2(0^+)] \\
& = -S2\hbar^2\gamma|\psi|^2\left\{\int_{-L}^{0} dx\phi_1''\delta\phi_1 + \int_{0}^{L} dx\phi_2''\delta\phi_2\right\} \\
& + \delta\phi_1(0^-)\left\{S2\hbar^2\gamma|\psi|^2\phi_1'(0^-) + F_J'[\phi_1(0^-) - \phi_2(0^+)]\right\} \\
& - \delta\phi_1(-L)\left\{S2\hbar^2\gamma|\psi|^2\phi_1'(-L)\right\} \\
& - \delta\phi_2(0^+)\left\{S2\hbar^2\gamma|\psi|^2\phi_2'(0^+) + F_J'[\phi_1(0^-) - \phi_2(0^+)]\right\} \\
& + \delta\phi_2(L)\left\{S2\hbar^2\gamma|\psi|^2\phi_2'(L)\right\}.
\end{aligned}$$

In equilibrium $\delta F = 0$. As usual, we fix $\delta\phi_1(-L) = 0$ and $\delta\phi_2(L) = 0$. Since the variation is arbitrary, the integrands must vanish. Thus $\phi_1'' = 0 = \phi_2''$. The variations at the origin are also arbitrary, and thus

$$\begin{aligned}
S2\hbar^2\gamma|\psi|^2\phi_1'(0^-) + F_J'[\phi_1(0^-) - \phi_2(0^+)] = 0 \\
S2\hbar^2\gamma|\psi|^2\phi_2'(0^+) + F_J'[\phi_1(0^-) - \phi_2(0^+)] = 0
\end{aligned}$$

Using the definition of the current density $j = 2q\hbar\gamma|\psi|^2\phi'$, these can be written as

$$Sj(0^-) = Sj(0^+) = -\frac{q}{\hbar}F_J'[\phi_1(0^-) - \phi_2(0^+)]$$

Using $F_J(\Delta\phi) = -E_J\cos\Delta\phi$, this becomes

$$Sj(0^-) = Sj(0^+) = -\frac{q}{\hbar}E_J\sin[\phi_1(0^-) - \phi_2(0^+)]$$

Defining $J = Sj(0^-) = Sj(0^+)$, $I_c = |q|E_J/\hbar$, and $\Delta\phi = \phi_1(0^-) - \phi_2(0^+)$, this is the desired result:

$$J = \frac{|q|}{\hbar}F_J'(\Delta\phi) = I_c\sin(\Delta\phi)$$

4. **DC SQUID**

Starting from the equations in the lectures

$$\begin{aligned}
&\Delta\phi_1 + \Delta\phi_2 = \frac{2\pi\Phi}{\Phi_0} + 2\pi N \\
&J = J_{c1}\sin(\Delta\phi_1) - J_{c2}\sin(\Delta\phi_2), \qquad (1)
\end{aligned}$$

show that for $J_{c1} = J_{c2} = J_c$ the current can be written

$$J = 2J_c(-1)^N \cos\frac{\pi\Phi}{\Phi_0} \sin\frac{\Delta\phi_1 - \Delta\phi_2}{2}. \tag{2}$$

Solution:
Since $J_{c1} = J_{c2} = J_c$, we have

$$J = J_c \left[\sin\left(\Delta\phi_1\right) - \sin\left(\Delta\phi_2\right)\right]. \tag{3}$$

Using the trigonometric identity

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) \tag{4}$$

we have

$$J = 2J_c \sin\left(\frac{\Delta\phi_1 - \Delta\phi_2}{2}\right)\cos\left(\frac{\Delta\phi_1 + \Delta\phi_2}{2}\right) \tag{5}$$

Because

$$\Delta\phi_1 + \Delta\phi_2 = \frac{2\pi\Phi}{\Phi_0} + 2\pi N, \tag{6}$$

we obtain

$$J = 2J_c \sin\left(\frac{\Delta\phi_1 - \Delta\phi_2}{2}\right)\cos\left(\frac{\pi\Phi}{\Phi_0} + \pi N\right). \tag{7}$$

Since

$$\cos\left(x + \pi N\right) = \cos\left(x\right)\cos\left(\pi N\right) - \sin\left(x\right)\sin\left(\pi N\right) = \left(-1\right)^N \cos(x), \tag{8}$$

we finally have

$$J = 2J_c \left(-1\right)^N \cos\left(\frac{\pi\Phi}{\Phi_0}\right)\sin\left(\frac{\Delta\phi_1 - \Delta\phi_2}{2}\right). \tag{9}$$