Statistical physics Final examination 13.12.2006
Suomalainen versio: käännä!

1. Show the following relations for ideal Bose-Einstein gas:
$$\begin{aligned}(\Delta n_\ell)^2 &\equiv \langle (n_\ell - \bar{n}_\ell)^2 \rangle = \bar{n}_\ell (1 + \bar{n}_\ell) \\ (\Delta N)^2 &\equiv \langle (N - \bar{N})^2 \rangle = \sum_\ell \bar{n}_\ell (1 + \bar{n}_\ell),\end{aligned}$$
and for ideal Fermi-Dirac gas:
$$(\Delta n_\ell)^2 = \bar{n}_\ell (1 - \bar{n}_\ell); \qquad (\Delta N)^2 = \sum_\ell \bar{n}_\ell (1 - \bar{n}_\ell).$$
Here $\bar{N} = \sum_\ell \bar{n}_\ell$ is the total number of particles.

2. Thermodynamic response functions can be obtained as second derivatives of thermodynamic potentials. Show that the heat capacities $C_V \equiv T \left(\frac{\partial S}{\partial T}\right)_{V,N}$ and $C_p \equiv T \left(\frac{\partial S}{\partial T}\right)_{p,N}$ can be expressed as
$$C_V = -T \left(\frac{\partial^2 F}{\partial T^2}\right)_{V,N}, \qquad C_p = -T \left(\frac{\partial^2 G}{\partial T^2}\right)_{p,N}.$$
Show also that the compressibilities $\kappa_T \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N}$ and $\kappa_S \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{S,N}$ can be written as
$$\kappa_T = -\frac{1}{V}\left(\frac{\partial^2 G}{\partial p^2}\right)_{T,N}, \qquad \kappa_S = -\frac{1}{V}\left(\frac{\partial^2 H}{\partial p^2}\right)_{S,N}.$$
Why the sign in front of the second derivative is negative?

3. Show that the expansion of the pressure for the ultrarelativistic Fermi-Dirac gas ($k_F \gg k_c = mc/\hbar$) at vanishing temperature $T = 0$ is
$$p = \frac{\hbar c}{12\pi^2}(k_F^4 - k_c^2 k_F^2 + \ldots)$$

4. Calculate the leading contribution to C_V when $T \ll T_F$ for ideal Fermi-Dirac gas. Hint: the 1st order Sommerfeld expansion is
$$\int_0^\infty d\epsilon\, \frac{\phi(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \approx \int_0^\mu d\epsilon\, \phi(\epsilon) + \frac{\pi^2}{6}(k_B T)^2 \phi'(\mu) + \ldots$$

5. The grand potential of the ideal Bose-Einstein gas is
$$\Omega = k_B T \ln(1 - z) - \frac{V k_B T}{\lambda_T^3} g_{5/2}(z), \qquad g_x(z) \equiv \sum_{n=1}^\infty \frac{z^n}{n^x}.$$
Here $\lambda_T^2 = 2\pi\hbar^2/(m k_B T)$ and z is the fugasity. Write down the Clausius-Clapeyron equation for the condensation transition and show that the condensation heat per particle is
$$\Delta h = \frac{5}{2}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} k_B T, \qquad \zeta(x) = g_x(1)$$