Statistical physics Problem set 13 (no excercise class)

NOTE: FINAL EXAM 13.12. 14 – 18 in TE320.

1. Ising model with the *mean field approximation*: Ising model partition function is
$$Z = \sum_{\{s_i\}} e^{\beta J \sum_{<ij>} s_i s_j + \beta h \sum_i s_i},$$
where $s_i = \pm 1$. One way to formulate the mean field approximation is to assume that we can substitute all of the neighbours of each spin s_i with the average spin $r \equiv \langle s \rangle$. Show that in this approximation we obtain the following condition for r:
$$r = \tanh[\beta(Jzr + h)],$$
where z is the coordination number (number of the nearest neighbours).

   Show that when $h = 0$, the critical temperature in this approximation is $T_c = Jz/k_B$. Also show that the critical exponent β, defined through
$$\langle r \rangle \propto (T_c - T)^\beta$$
when $T \lesssim T_c$, is $\beta = 1/2$.

   The above approximation gives a critical temperature also for 1-dim. Ising model, whereas the exact solution had none. Can you justify why?

2. *Landau's theory of phase transition* is formulated only in terms of the average order parameter. For example, in homogeneous and isotropic magnetic systems the Helmholtz free energy can be approximated by
$$F(T, m) = F_0(T) + \alpha(T - T_c)m^2 + \lambda m^4 + O(m^6),$$
where $F_0(T)$ is a regular function of T. Here $\vec{m} = \langle \vec{M} \rangle / V$, the average magnetisation. We shall neglect the $O(m^6)$ terms here. The Gibbs free energy in the presence of external field is
$$G(T, H) = F(T, m) - \mu_0 V \vec{m} \cdot \vec{H}$$

   a) When $\vec{H} = 0$, calculate $m^2(T)$. What is the order of the phase transition?

   b) Calculate the heat capacity
$$C_H = T \left(\frac{\partial S}{\partial T} \right)_H = -T \left(\frac{\partial^2 G}{\partial T^2} \right)_H$$
   when $H = 0$ both at $T < T_c$ and at $T > T_c$.

   c) Calculate the suscpetibility
$$\chi = \left(\frac{\partial M}{\partial H} \right)_T$$
   at $H = 0$. (hint: solve for $m(H, T)$ in the limit of small H.) How does χ behave in neighbourhood of T_c?

   d) What are the critical exponents β, γ, δ? (hint: you should obtain mean field results.)