Statistical physics Problem set 5 12.10.2006

1. Determine the trajectories of classical Hamiltonian flow in 2-dim. phase space (q,p), corresponding to a particle in a constant gravitational field

$$H = \frac{p^2}{2m} + mgq.$$

Calculate how a phase space area moves in time, if at time $t = 0$ it is a triangle with corners at $(q_0, p_0), (q_0 + a, p_0), (q_0, p_0 + b)$. Show that the surface area is conserved.

2. Baker's transformation ("fold and spread") is a simple form of mixing flow: it maps 2d unit square $0 \le x \le 1$, $0 \le y \le 1$ onto itself:

$$(x,y) \to (x',y') = U(x,y) = \begin{cases} (2x, y/2) & 0 \le x \le \frac{1}{2} \\ (2x-1, (y+1)/2) & \frac{1}{2} \le x \le 1. \end{cases}$$

   a) What is the inverse transform? Show that U preserves the area (hint: consider Jacobian of U). Now we can think

   $$(x,y) \to U(x,y) \to U^2(x,y) \to \ldots$$

   as being phase space flow in discrete time.

   b) Any x and y can be represented as binary numbers in form

   $$x = \sum_{n=-\infty}^{0} a_n 2^{n-1}, \qquad y = \sum_{n=1}^{\infty} a_n 2^{-n},$$

   where $a_n = 0$ or 1. Show that the Baker's transform is equivalent to so-called Bernoulli shift $a'_n = a_{n-1}$.

   c) Sketch how the area $0 \le x \le 1$, $0 \le y \le \frac{1}{2}$ transforms in a few consecutive transformations. How about a small square of size ϵ^2?
   Show that, if the center of a square of size ϵ^2 is at $x = \frac{1}{2}$, after $n \gtrsim -ln\epsilon/\ln 2$ transformations the square has been distributed to the whole phase space so that parts of it can be found within any square of size ϵ^2. How many transformations do you need to achieve this if the original square is at $x = 0$ boundary of the phase space? (deterministic irreversibility)

3. What can you say about a system with the following behaviour of the log of the density of states as a function of energy?