1. We have a gaussian probability distribution function

$$f(x) = Ce^{-\frac{1}{2}x^T g x}$$

$x = (x_1, x_2, \dots x_n)^T$, and g is $N \times N$ symmetric matrix.

   a) Show that $C = (2\pi)^{-n/2}\sqrt{\det g}$ (hint: diagonalize g)

   b) Show that

$$\langle x_p \cdots x_r \rangle = \left[\frac{\partial}{\partial h_p} \cdots \frac{\partial}{\partial h_r} F(h) \right]_{h=0}, \qquad F(h) = e^{\frac{1}{2}h^T g^{-1} h}$$

   c) Show that $\langle x_i x_j \rangle = (g^{-1})_{ij}$ and $\langle x_{i_1} x_{i_2} \dots x_{i_k} \rangle = 0$ if k odd

2. The density operator in canonical ensemble is $\rho = exp[-\beta(F - H)]$. Show that

$$\langle H \rangle = \frac{\partial(\beta F)}{\partial \beta} \quad \text{and} \quad \langle (H - \langle H \rangle)^3 \rangle = \frac{\partial^3(\beta F)}{\partial \beta^3}.$$

3. Let us assume that we have spin-$\frac{1}{2}$ particle with spin z-component σ_z eigenstates $|+\rangle$ and $|-\rangle$.

   a) In an experiment it has been verified that the probabilities of states $|\pm\rangle$ are $p_+ = \langle +|\rho|+\rangle$ and $p_- = \langle -|\rho|-\rangle = 1 - p_+$. Show that, with these conditions, the entropy is maximized by a density operator ρ with zero non-diagonal elements: $\langle +|\rho|-\rangle = \langle -|\rho|+\rangle = 0$. What is the maximum entropy?

   b) Let us assume that at time $t = 0$

$$\rho = \begin{bmatrix} p_+ & 0 \\ 0 & p_- \end{bmatrix}.$$

   The particle is set in a magnetic field along x-axis so that the Hamiltonian becomes $H = -h\sigma_x$. Determine $\rho(t)$ and the probability that the spin z-component state is $|+\rangle$. Determine $\langle \sigma_z \rangle$ and $(\Delta\sigma_z)^2 = \langle \sigma_z^2 \rangle - \langle \sigma_z \rangle^2$ as functions of time. (σ_i are 2×2 Pauli matrices.)