1. **Newton's law of gravity as a field equation**
According to Newton's law of gravity, the force on a body 1 (mass m_1) at $\boldsymbol{r}_1$ caused by a second body (mass m_2) at $\boldsymbol{r}_2$ is

$$\boldsymbol{F} = -G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}, \tag{1}$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and G is the constant of gravity. In this exercise we try to show that this implies that the acceleration $\boldsymbol{a}$ of a test particle at $\boldsymbol{r}$ in the presence of an arbitrary mass distribution with density $\rho(\boldsymbol{r})$ is given by

$$\boldsymbol{a}(\boldsymbol{r}) = -\boldsymbol{\nabla}V(\boldsymbol{r}), \tag{2}$$

where the potential V satisfies the Poisson equation

$$\nabla^2 V(\boldsymbol{r}) = 4\pi G\rho(\boldsymbol{r}). \tag{3}$$

a) Based on (1) and (2) show that the potential caused by one point particle of mass M at the origin corresponds to the potential

$$V(\boldsymbol{r}) = -G\frac{M}{r}. \tag{4}$$

b) Show that the potential (4) satisfies the Poisson equation for $\boldsymbol{r} \neq 0$.
c) By integrating the Poisson equation over a small sphere of radius ϵ around the origin, and transforming the left hand side to a surface integral, show that the potential (4) satisfies the Poisson equation also at $\boldsymbol{r} = 0$.
d) Based on the above, try to justify the claim in the introduction for an arbitrary mass distribution (not just one point mass).
You can use the following formulas appropriate for spherical coordinates (r, θ, ϕ):

$$\boldsymbol{\nabla}\Phi = \hat{\boldsymbol{r}}\frac{\partial\Phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} \tag{5}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}. \tag{6}$$

The motivation for this exercise is that equation (3) has some similarity with the corresponding equation in general relativity, that will be discussed later in this course.

2. **Four vector manipulation in special relativity**
Consider the four-vectors

$$\lambda^\mu = (2, 1, 1, 0) \text{ and } \sigma^\alpha = (1, 3, 0, 0). \tag{7}$$

a) Calculate λ_μ, σ_ν, $\lambda_\mu\lambda^\mu$, $\sigma^\alpha\sigma_\alpha$, $\sigma^\nu\lambda_\nu$.
b) Draw a sketch of the four vector λ^μ (7) in coordinate axes $(\lambda^1, \lambda^2, \lambda^0)$, where the λ^0 axis is drawn vertical. Can λ^μ represent a difference of events for the same particle? Do the same for σ^μ. Sketch also the surface $\lambda_\mu\lambda^\mu = 0$ (known as the light cone).
c) Consider the Lorentz-transformation $\lambda^{\mu'} = \Lambda^{\mu'}_{\nu}\lambda^\nu$, where

$$[\Lambda^{\mu'}_{\nu}] = \begin{bmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

and $\gamma = 1/\sqrt{1 - v^2/c^2}$. Show that when this is applied to event four vector $x^\mu = (ct, x, y, z)$, it is equivalent to the familiar form of the Lorentz transformation

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}. \end{aligned} \tag{9}$$

d) Using (7) and Lorentz transformation with $v = c/2$, calculate $\lambda^{\mu'}$, $\sigma^{\mu'}$ and $\lambda^{\mu'}\sigma_{\mu'}$. Compare the last one with $\sigma^\nu\lambda_\nu$.

1. **The coefficients of Lorentz transformation**
   Based on invariance of $s^2 = c^2t^2 - x^2 - y^2 - z^2$, show that in the transformation

$$\left.\begin{array}{l} t' = Bt + Cx \\ x' = A(x - vt) \\ y' = y \\ z' = z \end{array}\right\},$$

   the constants are $A = B = (1 - v^2/c^2)^{-1/2}$, $C = -(v/c^2)(1 - v^2/c^2)^{-1/2}$.

2. **Inverse Lorentz transformation**
   Equation

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -v\gamma/c & 0 & 0 \\ -v\gamma/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

   gives the matrix $[\Lambda^{\mu'}_{\nu}]$ for the boost in the x direction. What form does the inverse matrix $[\Lambda^{\nu}_{\mu'}]$ take? What is the velocity of K relative to K'?

3. **Newtonian limit of the Lorentz transformation**
   Show that when v/c is negligible, the equations of Lorentz boost:

$$t' = \gamma(t - xv/c^2),\ x' = \gamma(x - vt),\ y' = y,\ z' = z$$

   reduce to those of a Galilean boost:

$$t' = t,\ x' = x - vt,\ y' = y,\ z' = z.$$

4. **4-velocity**
   In a laboratory frame, write the 4-velocity u^μ for (a) a stationary chair, (b) a speeding bullet. Is it possible to write u^μ for a photon?

5. **Wave 4-vector**
   Consider an electromagnetic plane wave whose electric field is of the form

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{r}). \qquad (10)$$

   a) What is the frequency ν of the wave? How is $\boldsymbol{k}$ related to the propagation direction vector $\hat{\boldsymbol{n}}$ and the wave length λ of the wave? How should ω and $\boldsymbol{k}$ be related in order that the velocity of the wave be c?

b) We write the electric field of the wave in the form

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(k^\mu x_\mu). \tag{11}$$

Write $k^\mu x_\mu$ in terms of components and comparing with (10) identify the components of wave 4-vector k^μ. Express k^μ using λ and $\hat{\boldsymbol{n}}$.

6. **Electromagnetic field tensor**
   Show that the definitions

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{12}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \tag{13}$$

$$A^\mu = (\frac{\varphi}{c}, \boldsymbol{A}), \quad A_\mu = (\frac{\varphi}{c}, -\boldsymbol{A}) \tag{14}$$

and

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \tag{15}$$

lead to

$$[F_{\mu\nu}] = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}, \tag{16}$$

1. **Basis vectors of cylindrical coordinates**
   Cylindrical coordinates $(\rho,\, \phi,\, z)$ are defined by

$$\boldsymbol{r} = \rho\cos\phi\boldsymbol{i} + \rho\sin\phi\boldsymbol{j} + z\boldsymbol{k},$$

   where $0 \le \rho \le \infty$, $0 \le \phi \le 2\pi$ and $-\infty \le z \le \infty$. Obtain expressions for the natural basis vectors $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\phi$, $\boldsymbol{e}_z$ and the dual basis vectors $\boldsymbol{e}^\rho$, $\boldsymbol{e}^\phi$, $\boldsymbol{e}^z$ in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.

2. **Constant vector field in different coordinates**
   Show that, when referred to
   (a) the natural basis $\{\boldsymbol{e}_r,\, \boldsymbol{e}_\theta,\, \boldsymbol{e}_\phi\}$ of spherical coordinates,
   (b) the natural basis $\{\boldsymbol{e}_\rho,\, \boldsymbol{e}_\phi,\, \boldsymbol{e}_z\}$ of cylindrical coordinates,
   (c) the natural basis $\{\boldsymbol{e}_u,\, \boldsymbol{e}_v,\, \boldsymbol{e}_w\}$ of the paraboloidal coordinates $(u,\, v,\, w)$

$$x = u + v, \quad y = u - v, \quad z = 2uv + w,$$

   the constant vector field $\boldsymbol{i}$ is given by $(\operatorname{cosec}\theta = 1/\sin\theta)$
   (a) $\boldsymbol{i} = \sin\theta\cos\phi\boldsymbol{e}_r + r^{-1}\cos\theta\cos\phi\boldsymbol{e}_\theta - r^{-1}\operatorname{cosec}\theta\sin\phi\boldsymbol{e}_\phi$,
   (b) $\boldsymbol{i} = \cos\phi\boldsymbol{e}_\rho - \rho^{-1}\sin\phi\boldsymbol{e}_\phi$,
   (c) $\boldsymbol{i} = \frac{1}{2}\boldsymbol{e}_u + \frac{1}{2}\boldsymbol{e}_v - (u+v)\boldsymbol{e}_w$.

3. **Covariant componenets**
   Verify equation

$$\lambda_j = \boldsymbol{\lambda} \cdot \boldsymbol{e}_j,$$

   which shows that the covariant components λ_j of a vector $\boldsymbol{\lambda}$ are given by taking dot products of $\boldsymbol{\lambda}$ with the natural basis vectors $\boldsymbol{e}_j$.

4. **Transformation between basis vectors**
   Show that $\boldsymbol{e}_i = g_{ij}\boldsymbol{e}^j$ and $\boldsymbol{e}^i = g^{ij}\boldsymbol{e}_j$.

5. **Simplifications**
   Simplify the following expressions:
   (a) $\lambda^i\delta_i^j\lambda_j$, (b) $\mu_i g^{ij} g_{jk}\lambda^k$, (c) $g_{ij}\lambda^i\mu^j - \lambda^k\mu_k$.

6. **G and $\hat{G}$ in orthogonal coordinates**
   If the coordinate system is orthogonal, what can you say about the matrices $G \equiv [g_{ij}]$ and $\hat{G} \equiv [g^{ij}]$?

7. G **and** $\hat{G}$ **in paraboloidal coordinates**
   Inverting the paraboloidal coordinates of Problem 2c gives

$$u = \frac{1}{2}(x + y), \quad v = \frac{1}{2}(x - y), \quad w = z - \frac{1}{2}(x^2 - y^2).$$

   Form the natural and dual basis vectors and show that they lead to

$$G = [g_{ij}] = \begin{bmatrix} 2(1 + 2v^2) & 4uv & 2v \\ 4uv & 2(1 + 2u^2) & 2u \\ 2v & 2u & 1 \end{bmatrix},$$

$$\hat{G} = [g^{ij}] = \begin{bmatrix} 1/2 & 0 & -v \\ 0 & 1/2 & -u \\ -v & -u & 2u^2 + 2v^2 + 1 \end{bmatrix}.$$

   Show that these satisfy $G\hat{G} = \hat{G}G = I$.

8. **Contravariant components in paraboloidal coordinates**
   In paraboloidal coordinates (Problems 2c and 7) a vector field $\boldsymbol{\mu}$ has covariant components given by

$$\mu_i = v\delta_i^1 - u\delta_i^2 + \delta_i^3.$$

   What are its contravariant components μ^i?

9. **Kronecker delta**
   A repeated suffix implies summation. What, then are the values of
   (a) δ_i^i, (b) δ_A^A, (c) δ_a^a, (d) δ_μ^μ?

   Remember that uppercase literal suffixes A, B, ... are used when referring to a two dimensional space, and take values 1 and 2; suffixes i, j, k, ... take values 1, 2, 3; Greek suffixes take values 0, 1, 2, 3; lower case letters from the beginning of the alphabet refer to an N-dimensional space, and have the range 1, 2, 3, ..., N.

1. **Line element**
For the paraboloidal coordinates (Example 1.2.1 in FN)

$$G = [g_{ij}] = \begin{bmatrix} 2(1+2v^2) & 4uv & 2v \\ 4uv & 2(1+2u^2) & 2u \\ 2v & 2u & 1 \end{bmatrix}.$$

What form does the line element $ds^2 = g_{ij} du^i du^j$ take in these coordinates?

2. **Curve length**
Describe the curve given in cylindrical coordinates by

$$\rho = a,\ \phi = t,\ z = t,\ -\pi \leq t \leq \pi$$

(where a is a positive constant) and find its length.

3. **Arc length as parameter**
Suppose that in the curve $\boldsymbol{r}(s)$, the arc length s (measured along a curve from some base point) is used as a parameter. Calculate the length of the tangent vector $\dot{\boldsymbol{r}}(s)$ and argue that it is equal to 1.

4. **Relationship of $U^j_{i'}$ and $U^{j'}_i$**
Use the chain rule to show that $U^k_{i'} U^{i'}_j = \delta^k_j$ and $U^{k'}_i U^i_{j'} = \delta^k_j$. Obtain the same results by using the fact that $\delta^k_j = \boldsymbol{e}^k \cdot \boldsymbol{e}_j = \boldsymbol{e}^{k'} \cdot \boldsymbol{e}_{j'}$.

5. **Transformation of covariant components**
Obtain the equation $\mu_i = U^{j'}_i \mu_{j'}$ using equation $\mu_{i'} = U^j_{i'} \mu_j$ and the result of exercise 4.

6. **Transformation of g_{ij} in matrix form**
Translate equation

$$g_{i'j'} = U^k_{i'} U^l_{j'} g_{kl}$$

into a matrix equation involving

$$\hat{U} \equiv [U^i_{j'}],\ G \equiv [g_{ij}],\ G' \equiv [g_{i'j'}].$$

Hence, use G in spherical coordinates (Example 1.3.1 in FN)

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

and the transformation matrix between cylindrical and spherical coordinates (Example 1.4.1 in FN)

$$\hat{U} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ (\cos\theta)/r & 0 & -(\sin\theta)/r \\ 0 & 1 & 0 \end{bmatrix},$$

to obtain the line element for Euclidean space in cylindrical coordinates.

7. **Transformation of the stress tensor**
   Show that the components τ^i_j of the stress tensor $\boldsymbol{\tau}$ are given by

$$\tau^i_j = \boldsymbol{e}^i \cdot \boldsymbol{\tau}(\boldsymbol{e}_j)$$

   and use this result to re-establish the transformation formula

$$\tau^{i'}_{m'} = U^{i'}_k U^l_{m'} \tau^k_l.$$

1. **Basis vectors of the tangent plane**
Starting from $\boldsymbol{r} = (u+v)\boldsymbol{i} + (u-v)\boldsymbol{j} + 2uv\boldsymbol{k}$, calculate the natural basis $\{\boldsymbol{e}_A\} = \{\boldsymbol{e}_u, \boldsymbol{e}_v\}$, and the quantities g_{AB} and g^{AB}. (Hint: you can use the identity $G\hat{G} = I$.)

2. **Line element of a surface**
Write down the line element for
(a) a sphere of radius a, using angles (θ, ϕ) borrowed from spherical coordinates as parameters;
(b) a cylinder whose cross section is a circle of radius a, using (θ, z) borrowed from cylindrical coordinates as parameters;
(c) the hyperbolic paraboloid of exercise 1, using the parameters (u, v) of that exercise.

3. **Flatness of a surface**
Is the cylinder of exercise 2(b) curved or flat? By flat, one means that the line element can be written as $ds^2 = du^2 + dv^2$ in some coordinates (u, v).

4. **Coordinate dependence of $\tau_{ab} = \delta_{ab}$**
Suppose that in some coordinate system the components τ_{ab} of a type (0, 2) tensor satisfy $\tau_{ab} = \delta_{ab}$. Show that this property is *not* coordinate-independent.
(Use the transformations between spherical and cylindrical coordinates developed in exercise 4.6 as the basis for a counter example.)

5. **Symmetric tensor**
Verify that the relationship $\tau^{ab} = \tau^{ba}$, defining a symmetric tensor, is coordinate-independent.

6. **A tensor identity**
Show that if $\sigma_{ab} = \sigma_{ba}$ and $\tau^{ab} = -\tau^{ba}$ for all a, b, then $\sigma_{ab}\tau^{ab} = 0$.

7. **Decomposition of a tensor into symmetric and antisymmetric parts**
Show that any type (2, 0) or type (0, 2) tensor can be expressed as the sum of a symmetric and an antisymmetric ($\tau^{ab} = -\tau^{ba}$) tensor.

Continues...

8. **Coordinate transformation**
Show that if at a point P of a manifold the contravariant vector λ^a is nonzero, then it is possible to change to a new (primed) coordinate system in which $\lambda^{a'} = \delta^a_1$ at the P.
(A simple transformation between coordinates $x^{a'}$ and x^b could be $x^{a'} = A^{a'}_b x^b$ with a *constant* matrix $A = [A^{a'}_b]$. Assuming this form write the conditions that A should satisfy, and argue that they all can be satisfied.)

9. **Another tensor identity**
If τ^{ab} is a symmetric tensor and λ^a a contravariant vector with the property that

$$\tau^{bc}\lambda^a + \tau^{ca}\lambda^b + \tau^{ab}\lambda^c = 0$$

for all a, b, c, deduce that either $\tau^{ab} = 0$ or $\lambda^a = 0$.
(Hint: If at the point in question $\lambda^a \neq 0$, then we can introduce the special coordinate system of previous exercise.)

1. **Parameter independence of arc length**
Show that the definition of the length of a curve given by equation

$$L = \int_{t_a}^{t_b} |g_{ab}\dot{x}^a\dot{x}^b|^{1/2} dt$$

is independent of the parameter used.

2. **Vectors in Schwarzschild metric**
For $r > 2m$, the Schwarzschild solution has a metric tensor field given by

$$[g_{\mu\nu}] = \text{diag}(c^2(1-2m/r), -(1-2m/r)^{-1}, -r^2, -r^2\sin^2\theta),$$

where the coordinates are labeled according to $t \equiv x^0$, $r \equiv x^1$, $\theta \equiv x^2$, $\phi \equiv x^3$. Find the lengths of the following vectors and the angles between them:

$$\text{(a) } \lambda^\mu \equiv \delta^\mu_0;\ \text{(b) } \mu^\mu \equiv \delta^\mu_1;\ \text{(c) } \nu^\mu \equiv \delta^\mu_0 + c(1-2m/r)\delta^\mu_1.$$

Are any of these vectors null? Are any pairs orthogonal?

3. **Coordinate transformation**
Let x^i be a system of Cartesian coordinates in Euclidean space, and let $x^{i'}$ be a new system whose axes are obtained by rotating those of the unprimed system about its x^3 axis through an angle θ in the positive sense.

a) Show that at each point of space the new basis vectors are given in terms of the old basis vectors by

$$\boldsymbol{e}_{1'} = \cos\theta\,\boldsymbol{e}_1 + \sin\theta\,\boldsymbol{e}_2, \quad \boldsymbol{e}_{2'} = -\sin\theta\,\boldsymbol{e}_1 + \cos\theta\,\boldsymbol{e}_2, \quad \boldsymbol{e}_{3'} = \boldsymbol{e}_3.$$

What are the transformation matrices $[X^{i'}_j]$ and $[X^i_{j'}]$?

b) Recall that, for a rigid body having one of its points fixed at the origin O, its angular momentum L^i about O can be expressed as $L^i = I^i_j\omega^j$, where I^i_j is the *inertia tensor* of the body about O and ω^i is its angular momentum (all regarded as tensors at O). Find $[L^i]$ when

$$[I^i_j] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \text{ and } [\omega^i] = \begin{bmatrix} 0 \\ 15 \\ 0 \end{bmatrix}$$

c) Transform the components to find $I^{i'}_{j'}$, $\omega^{i'}$, and $L^{i'}$ relative to the new coordinate system, and check that $L^{i'} = I^{i'}_{j'}\omega^{j'}$.

4. **General parameter in geodesic equation**
Show that if a general parameter $t = t(s)$ (where $dt/ds \neq 0$) is used to parameterize a straight line in Euclidean space, then the geodesic equation takes the form

$$\frac{d^2u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = h(s)\frac{du^i}{dt} \text{ where } h(s) = -\frac{d^2t}{ds^2}\left(\frac{dt}{ds}\right)^{-2}.$$

Deduce that this reduces to the simple form

$$\frac{d^2u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = 0$$

if, and only if, $t = As + B$, where A, B are constants $(A \neq 0)$.

5. **Lenght of tangent vector**
The aim of this exercise is to show that the length L of the tangent vector $\dot{x}^a$ to an affinely parameterized (i.e. the parameter is of the form $t = As + B$) geodesic is constant.

a) Start by arguing that $\pm L^2 = g_{ab}\dot{x}^a\dot{x}^b$.

b) Differentiate this equation to obtain an expression for $\pm 2L\dot{L}$ in terms of quantities g_{ab}, $\dot{g}_{ab}$, $\dot{x}^a$, and $\ddot{x}^a$.

c) Put $\dot{g}_{ab} = \partial_c g_{ab}\dot{x}^c$ and use the geodesic equation $\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$ to express the second derivates $\ddot{x}^a$ in term of the connection coefficients Γ^a_{bc} and the first derivates $\dot{x}^a$.

d) Then use equation $\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$ to express the Γ^a_{bc} in terms of the metric tensor components and their derivates.

e) Simplify to obtain $2L\dot{L} = 0$, from which it follows that $\dot{L} = 0$ and L is constant.

1. **Connection coefficients for a spherical surface**
In exercise 5.2(a) it was shown that the line element on a sphere of radius a using spherical coordinates is $ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta\, d\phi^2$. Defining the coordinates $u^1 \equiv \theta$, $u^2 \equiv \phi$, show that the metric tensor components are given by

$$[g_{AB}] = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{bmatrix}.$$

Deduce using the Lagrangian method that the only nonzero connection coefficients are

$$\Gamma^1_{22} = -\sin\theta\cos\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta.$$

2. **Geodesics for a spherical surface**
Show that all lines of longitude on a sphere (curves given by $\phi =$ constant) are geodesics.

3. **Time-like geodesics in Robertson-Walker spacetime**
Robertson-Walker spacetime is defined by the line element $g_{\mu\nu}dx^\mu dx^\nu = dt^2 - [R(t)]^2[(1-kr^2)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2]$, where $\mu,\nu = 0,1,2,3$ and $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \phi$. A coordinate curve for which r, θ, ϕ are constant and t varies is given by

$$x^\mu(u) = u\delta^\mu_0 + r_0\delta^\mu_1 + \theta_0\delta^\mu_2 + \phi_0\delta^\mu_3,$$

where r_0, θ_0, ϕ_0 are constants and u is a parameter. Verify that all such coordinate curves are geodesics affinely parameterized by u.

4. **Parallel transport on a spherical surface**
Consider a sphere of radius a, with coordinates $u^1 \equiv \theta$, $u^2 \equiv \phi$ borrowed from spherical coordinates, where $0 < \theta < \pi$ and $0 < \phi < 2\pi$. Let us transport a vector $\boldsymbol{\lambda}$ parallelly around the circle of latitude γ given by $\theta = \theta_0$ ($\theta_0 =$ constant), starting and ending at the point P_0 where $\phi = 0$ or 2π. The circle is given parametrically by

$$u^A = \theta\delta^A_1 + t\delta^A_2,\ 0 < t < 2\pi,$$

so $\dot{u} = \delta^A_2$ and the equation for parallel transport becomes $\dot{\lambda}^A + \Gamma^A_{B2}\lambda^B = 0$. Verify that the initial-value problem comprising the pair of equations

$$\begin{cases} \dot{\lambda}^1 - \sin\theta_0\cos\theta_0\lambda^2 = 0 \\ \dot{\lambda}^2 + \cot\theta_0\lambda^1 = 0 \end{cases}$$

with initial conditions

$$\begin{cases} \lambda^1(0) = a^{-1} \cos\alpha \\ \lambda^2(0) = (a \sin\theta_0)^{-1} \sin\alpha \end{cases}$$

has a solution given by equations

$$\begin{cases} \lambda^1 = a^{-1} \cos(\alpha - \omega t) \\ \lambda^2 = (a \sin\theta_0)^{-1} \sin(\alpha - \omega t), \end{cases}$$

where $\omega = \cos\theta_0$.

5. **Reversal in transport on a closed path**
   For what circle(s) of latitude is the final direction of the transported vector in Exercise 7.4 exactly opposite to that of the initial direction?

6. **Angle between vectors in parallel transport**
   Noting that the equator ($\theta_0 = \pi/2$) is a geodesic and has tangent vector $\mu^A \equiv a^{-1}\delta^A_2$, verify that for parallel transport along a geodesic the angle between the transported vector of Exercise 7.4 and the tangent to the geodesic is constant.

1. **Transformation of Γ^a_{bc}**
   Show that an alternative form for the transformation formula
   $$\Gamma^{a'}_{b'c'} = \Gamma^d_{ef} X^{a'}_d X^e_{b'} X^f_{c'} + X^d_{c'b'} X^{a'}_d$$
   is
   $$\Gamma^{a'}_{b'c'} = \Gamma^d_{ef} X^{a'}_d X^e_{b'} X^f_{c'} - X^e_{b'} X^f_{c'} X^{a'}_{ef}.$$

2. **Transformation of absolute derivative**
   By using the transformation rules to the quantities on the left hand side, verify the formula
   $$\dot{\lambda}^{a'} + \Gamma^{a'}_{b'c'} \lambda^{b'} \dot{x}^{c'} = X^{a'}_d \left(\dot{\lambda}^d + \Gamma^d_{ef} \lambda^e \dot{x}^f \right).$$
   Based on this, deduce that the defining equation for parallel transport of a contravariant vector along a curve
   $$\dot{\lambda}^a + \Gamma^a_{bc} \lambda^b \dot{x}^c = 0$$
   is coordinate-independent.

3. **Absolute derivative of (0,2) and (1,1) tensors**
   Obtain formulae
   $$\frac{D\tau_{ab}}{du} \equiv \dot{\tau}_{ab} - \Gamma^c_{ad} \tau_{cb} \dot{x}^d - \Gamma^c_{bd} \tau_{ac} \dot{x}^d \text{ and } \frac{D\tau^a_b}{du} \equiv \dot{\tau}^a_b + \Gamma^a_{cd} \tau^c_b \dot{x}^d - \Gamma^c_{bd} \tau^a_c \dot{x}^d,$$
   using similar method as writing $\tau^{ab} = \lambda^a \mu^b$ in deriving the result
   $$\frac{D\tau^{ab}}{du} \equiv \dot{\tau}^{ab} + \Gamma^a_{cd} \tau^{cb} \dot{x}^d + \Gamma^b_{cd} \tau^{ac} \dot{x}^d.$$

4. **Geodesic equation using absolute derivative**
   Show that equation
   $$\frac{d^2 x^a}{du^2} + \Gamma^a_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = 0$$
   for an affinely parameterized geodesic can be written as
   $$\frac{D\dot{x}^a}{du} = 0.$$

5. **Length of tangent vector using absolute derivative**
   Prove that the length of the tangent vector $\dot{x}^a$ to an affinely parameterized geodesic is constant.

1. **Geodesic coordinates**
Show that, as a result of the coordinate transformation leading to geodesic coordinates [equation $x^{a'} \equiv x^a - x^a_O + \frac{1}{2}(\Gamma^a_{bc})_O(x^b - x^b_O)(x^c - x^c_O)$], we can write

$$(g_{a'b'})_O = (g_{ab})_O.$$

2. **Equation of motion for a free particle**
Deduce the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

from the equations

$$Dp^\mu/d\tau = f^\mu, \quad p^\mu \equiv mu^\mu,$$

in the case of a free particle, for which $f^\mu = 0$.

3. **Derivatives of inverse functions**
Given the function $\tau(t)$, the derivative of the inverse function $t(\tau)$ is

$$\frac{dt}{d\tau} = \left(\frac{d\tau}{dt}\right)^{-1}.$$

Show that the second derivative is

$$\frac{d^2t}{d\tau^2} = -\frac{d^2\tau}{dt^2}\left(\frac{d\tau}{dt}\right)^{-3}.$$

Use this result to show that

$$h(t) = -\frac{d^2t}{d\tau^2}\left(\frac{dt}{d\tau}\right)^{-2} = \frac{d^2\tau}{dt^2}\left(\frac{d\tau}{dt}\right)^{-1}. \tag{17}$$

4. **Newtonian limit**
(a) Assume that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ are small.
By writing $g^{\mu\nu} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu}$, where $\tilde{h}^{\mu\nu}$ are small, show that

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\sigma}\eta^{\nu\rho}h_{\sigma\rho}. \tag{18}$$

(b) Show that for nonrelativistic velocities,

$$\left(\frac{d\tau}{dt}\right)^2 = 1 + h_{00}.$$

Hence deduce that the term on the right hand side in the geodesic equation (Exercise 6.4)

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\nu\sigma}\frac{dx^\nu}{dt}\frac{dx^\sigma}{dt} = h(t)\frac{dx^i}{dt},$$

where $h(t)$ is given in Eq. (17), is unimportant in the Newtonian limit, where $h_{00} = 2V/c^2$.

5. **Rotating coordinates**
   Starting from the line element

$$c^2d\tau^2 = c^2dT^2 - dX^2 - dY^2 - dZ^2$$

and the transformation

$$\begin{cases} T = t \\ X = x\cos\omega t - y\sin\omega t \\ Y = x\sin\omega t + y\cos\omega t \\ Z = z \end{cases}$$

check that the line element in coordinates (t, x, y, z) is given by

$$c^2d\tau^2 = \left[c^2 - \omega^2\left(x^2 + y^2\right)\right]dt^2 + 2\omega y\,dx\,dt - 2\omega x\,dy\,dt - dx^2 - dy^2 - dz^2.$$

6. **Constant gravitational field**
   The potential $V = gz$ of constant gravitational field correspons to the line element

$$c^2d\tau^2 = c^2\left(1 + \frac{2gz}{c^2}\right)dt^2 - dx^2 - dy^2 - dz^2.$$

Use the Lagrange method to calculate the connection coefficients $\Gamma^\mu_{\nu\sigma}$, using the variables $x^0 = t$, $x^1 = x$, $x^2 = y$, and $x^3 = z$. Show that in the limit $c \to \infty$, the geodesics satisfy the Newtonian equations of motion

$$m\frac{d^2\boldsymbol{r}}{dt^2} = -mg\boldsymbol{e}_3.$$

1. **Static spherically symmetric spacetime**
The line element of a static spherically symmetric spacetime is

$$c^2 d\tau^2 = A(r)\, dt^2 - B(r)\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2.$$

Use the Euler-Lagrange equations to obtain the geodesic equations, and hence show that the only nonvanishing connection coefficients are:

$$\begin{array}{lll} \Gamma^0_{01} = \Gamma^0_{10} = A'/2A, & \Gamma^1_{00} = A'/2B, & \Gamma^1_{11} = B'/2B, \\ \Gamma^1_{22} = -r/B, & \Gamma^1_{33} = -(r\, \sin^2\theta)/B, & \Gamma^2_{12} = \Gamma^2_{21} = 1/r, \\ \Gamma^2_{33} = -\sin\theta\cos\theta, & \Gamma^3_{13} = \Gamma^3_{31} = 1/r, & \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \end{array}$$

where primes denote derivatives with respect to r, and

$$x^0 \equiv t,\, x^1 \equiv r,\, x^2 \equiv \theta,\, x^3 \equiv \phi.$$

2. **Energy-momentum-stress tensor for fluid at rest**
Show that in Cartesian coordinate system, which brings the velocity of the fluid at a point P to rest (i.e., in an instantaneous rest system for the fluid at P), the components of the stress tensor (as defined by $T^{\mu\nu} \equiv (\rho + p/c^2)u^\mu u^\nu - p\eta^{\mu\nu}$) are given by

$$[T^{\mu\nu}] = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}.$$

3. **Dimensional consistency of the energy-momentum-stress tensor**
What is the unit of $T^{\mu\nu}$? Check that all the terms on the right-hand side of equation $T^{\mu\nu} \equiv (\rho + p/c^2)u^\mu u^\nu - p\eta^{\mu\nu}$ have the same units.

4. **A simple indentity**
Verify that $u^\mu u_\mu = c^2$ implies that $u^\mu{}_{;\nu} u_\mu = 0$.

5. **Curvature tensor**
(a) Show that for a contravariant vector field λ^a,

$$\lambda^a{}_{;bc} - \lambda^a{}_{;cb} = -R^a{}_{dbc}\lambda^d.$$

(b) Show that for a type (2, 0) tensor field τ^{ab},

$$\tau^{ab}{}_{;cd} - \tau^{ab}{}_{;dc} = -R^a{}_{ecd}\tau^{eb} - R^b{}_{ecd}\tau^{ae}.$$

(Without loss of generality take $\tau^{ab} = \lambda^a \mu^b$.)

(c) Guess the corresponding expression for a type (2, 1) tensor field τ^{ab}_c.

6. **Cyclic identity**
Prove the cyclic identity $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$.

7. **Symmetry of Ricci tensor**
By contracting the cyclic identity $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$, prove that the Ricci tensor, $R_{ab} = R^c{}_{abc}$, is symmetric.

1. **Alternative form of Einstein's field equation**
   By contracting the mixed form $R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu = \kappa T^\mu_\nu$ of equation $R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \kappa T^{\mu\nu}$ show that $R = -\kappa T$, where $T \equiv T^\mu_\mu$, and hence verify equation $R^{\mu\nu} = \kappa(T^{\mu\nu} - \frac{1}{2}Tg^{\mu\nu})$.

2. **Ricci tensor in a static spherically symmetric spacetime**
   We have
   $$R_{\mu\nu} \equiv \partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma}$$
   and from Exercise 9.1 we have
   $$\begin{array}{lll} \Gamma^0_{01} = \Gamma^0_{10} = A'/2A, & \Gamma^1_{00} = A'/2B, & \Gamma^1_{11} = B'/2B, \\ \Gamma^1_{22} = -r/B, & \Gamma^1_{33} = -(r\,\sin^2\theta)/B, & \Gamma^2_{12} = \Gamma^2_{21} = 1/r, \\ \Gamma^2_{33} = -\sin\theta\cos\theta, & \Gamma^3_{13} = \Gamma^3_{31} = 1/r, & \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \end{array}$$
   show that
   $$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}$$
   and $R_{0i} = 0$ $(i = 1, 2, 3)$.

3. **Curvature tensor of a two-dimensional manifold**
   Show that in a two-dimensional Riemannian manifold all components of R_{ABCD} are either zero or $\pm R_{1212}$. In terms of the usual spherical coordinates $u^1 \equiv \theta$ and $u^2 \equiv \phi$, the metric tensor field of a sphere of radius a is given by
   $$[g_{AB}] = \begin{bmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{bmatrix}.$$
   Show that $R_{1212} = a^2\sin^2\theta$, and hence deduce that
   $$[R_{AB}] = \begin{bmatrix} -1 & 0 \\ 0 & -\sin^2\theta \end{bmatrix}$$
   and $R = -2/a^2$.

4. **Isotropic form of Schwarzschild metric**
   Show that the Schwarzschild line element
   $$c^2d\tau^2 = \left(1 - \frac{2m}{r}\right)c^2dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2,$$

where $m = GM/c^2$, may be put into the *isotropic form*

$$\begin{aligned} c^2 d\tau^2 &= \left(1 - \frac{m}{2\rho}\right)^2 \left(1 + \frac{m}{2\rho}\right)^{-2} c^2 dt^2 \\ &\quad - \left(1 + \frac{m}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2\right), \end{aligned}$$

where the new coordinate ρ is defined by

$$r \equiv \rho \left(1 + \frac{m}{2\rho}\right)^2$$

1. **Radial distance in Schwarzschild metric**
Show that the length of stick laying radially (between r_1 and $r_2 > r_1$) in Schwarzschild metric is

$$L = \int_{r_1}^{r_2} \left(1 - \frac{2m}{r}\right)^{-1/2} dr = r_2 - r_1 + m \ln \frac{r_2}{r_1} + O(m^2/r),$$

where $m = GM/c^2$.

2. **Spectral shift**
Find the fractional shift in frequency, as measured on Earth, for light from a star of mass 10^{30} kg, assuming that the photons come from just above the star's atmosphere where $r_B = 1000$ km.

3. **Particle motion in Schwarzschild metric, part I**
With the variables $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and with w as an affine parameter, obtain the second and third ($\mu = 1$ and $\mu = 2$) of equations

$$\frac{d}{dw}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0,$$

where

$$L(\dot{x}^\sigma, x^\sigma) \equiv \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left[(1 - 2m/r)c^2 \dot{t}^2 - (1 - 2m/r)^{-1} \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)\right].$$

Hence show that $\theta = \pi/2$ satisfies the third equation, and with $\theta = \pi/2$ the second equation reduces to

$$(1 - 2m/r)^{-1} \ddot{r} + \frac{mc^2}{r^2} \dot{t}^2 - (1 - 2m/r)^{-2} \frac{m}{r^2} \dot{r}^2 - r\dot{\phi}^2 = 0.$$

4. **Particle motion in Schwarzschild metric, part II**
Continuing the preceding exercise, show that the first and fourth equations ($\mu = 0$ and $\mu = 3$) give the conditions

$$\left(1 - \frac{2m}{r}\right) \dot{t} = \text{constant} = k, \qquad r^2 \dot{\phi} = \text{constant} = h. \tag{19}$$

In addition we have the condition

$$c^2 = \left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2$$

for a particle using the proper time τ as the parameter w. Based on these, check the equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = E + \frac{2GM}{h^2}u + \frac{2GM}{c^2}u^3,$$

where $E \equiv c^2(k^2-1)/h^2$ and $u \equiv 1/r$.

5. **Eddington-Finkelstein coordinates**
   Verify the form

$$c^2d\tau^2 = (1-2m/r)dv^2 - 2\,dv\,dr - r^2d\theta^2 - r^2\sin^2\theta\,d\phi^2$$

   of the line element in Eddington-Finkelstein coordinates by replacing the Schwarzschild coordinate t by $v = ct + r + 2m\ln(r/2m - 1)$.

6. **Schwarzschild radius**
   Find the Schwarzschild radius of a spherical object with the same mass as that of the Earth. (Take $M_\oplus = 6\times 10^{24}$ kg, $G = 6.67\times 10^{-11}$ Nm^{2}kg^{-2}, $c = 3\times 10^8$ ms^{-1}.)